State-Dependent Symbol-Wise Decode and Forward Codes Over Multihop Relay Networks

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***Abstract*— This paper studies low-latency streaming codes for the multi-hop network. The source transmits a sequence of messages to a destination through a chain of relays, and requires the destination to reconstruct each message by its deadline. We assume that each communication link is subjected to a certain maximum number of packet erasures. The case of a single relay (a three-node network) was considered in Fong *et al.* (2020). A coding scheme known as symbol-wise decode and forward was proposed. In the present work, we propose an alternative scheme that is different from Fong *et al.* (2020) and still achieves the same rate as in Fong *et al.* (2020) for the one hop case as the field-size goes to infinity. Furthermore, our proposed scheme naturally generalizes to the case of multiple-relay nodes yielding new achievable rates for this setting. The main difference with Fong *et al.* (2020) is that our proposed scheme exploits the ability of the relay nodes to adapt the transmission based on the erasures on the previous link. Hence, we refer to our scheme as "state-dependent" and contrast it with the scheme in Fong *et al.* (2020) that is state-independent. Our scheme requires the relay nodes to append a header to the transmitted packets, and we show that the size of the header does not depend on the field-size of the code. We also derive an upper bound on the maximal streaming rate achievable over a network with an arbitrary number of relays. We show that this upper bound matches our achievable rate in the special case when the maximal number of erasures on the first link is greater than or equal to the maximal number of erasures on each of the following links, and the field size goes to infinity.**

***Index Terms*— Streaming codes, delays, forward error correction (FEC).**

I. Introduction

REAL-TIME interactive video streaming is an integral part of the day-to-day activity of many people in the world. Traditionally, most of the traffic on the internet is not sensitive to the typical delay induced by the network. However, as networks evolved, more and more people are using the network for real-time conversations, video conferencing, and online monitoring. According to [2], IP video traffic will account for 82 percent of traffic by 2022. Further, live video is projected to grow 15-fold to reach 17 percent of Internet video traffic by 2022.

All types of traffic are susceptible to errors, and therefore many applications use an error-correcting mechanism. One fundamental difference between real-time video streaming and other types of traffic is the (much more stringent) latency requirement each packet has to meet in order to provide a good user experience. A very common error-correcting mechanism is automatic repeat request (ARQ). Using ARQ means that the latency (in case of an error) is at least three times the one-way delay, which in many cases may violate the latency requirements for real-time interactive video streaming.

An alternative method for handling errors in the transmission is forward error correction (FEC). Using FEC has the potential to lower the recovery latency since it does not require communication between the receiver and transmitter. However, in many cases, when FEC is designed, the emphasis is on its error-correcting capabilities while ignoring latency constraints. Two commonly used codes are Low-density parity-check (LDPC) [3], [4] and digital fountain codes [5], [6]. The typical block length of these codes is very long (usually a few hundreds of symbols) hence precluding their use for real-time interactive applications.

Low-latency FEC codes are already implemented and have a noticeable impact on the quality of real-time interactive applications. Typically, maximum-distance separable (MDS) codes are used to transmit an extra parity-check packet per every two to five packets [7]. For example, in [8], the FEC implemented in Skype is described, and it is argued that this mechanism is one of the main contributors to the success of this application.

In this paper we build upon the line of work initiated in [9], where a class of low-latency FEC (referred to as streaming codes) was introduced and shown to achieve optimal error correction for a class of burst-erasure channels. This work was followed by a plurality of works [10]–[16] which extended the channel model to contain both bursts and arbitrary erasures. However, all these works focussed on a point-to-point setting. The performance of streaming codes for the three-node network was studied in [1]. The authors propose a new coding scheme denoted as "symbol-wise" decode and forward (SWDF) in which the relay forwards

Manuscript received 29 April 2021; revised 3 April 2022; accepted 16 June 2022. Date of publication 4 July 2022; date of current version 21 October 2022. An earlier version of this paper was presented in part at the 2020 IEEE International Symposium on Information Theory [DOI: 10.1109/ISIT44484.2020.9174225]. *(Corresponding author: Elad Domanovitz.)*

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Communicated by A. Thangaraj, Associate Editor for Coding and Decoding.

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TIT.2022.3187669.

Digital Object Identifier 10.1109/TIT.2022.3187669

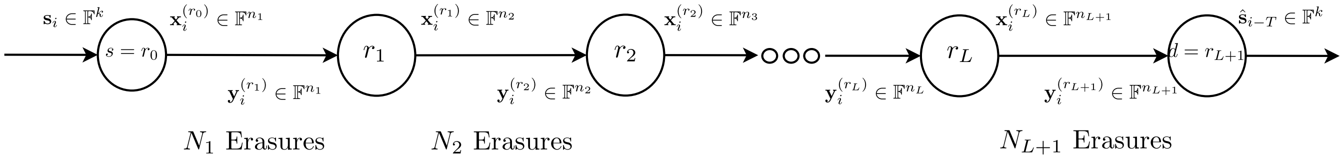

Fig. 1. Symbols generated in the L-node relay network at time i.

the recovered symbols (before the entire message can be decoded) to the destination in a carefully selected manner. The authors demonstrated that SWDF is superior to a naive "message-wise" decode and forward strategy, where each message must be fully decoded at the relay prior to forwarding it. Unfortunately, this coding scheme does not easily scale beyond the three-node relay network. A key property of the coding scheme described in [1], is "state-independence", i.e., it does not depend on the specific location of erasures in the different segments. In this paper, we propose an alternative scheme that can recover the achievable rate in [1] and also can be generalized to any number of relays. Our proposed scheme is a "state-dependent" scheme, i.e., it is a scheme that reorders the symbols transmitted by each relay based on the erasure patterns that occurred in the previous links. As a result, our proposed scheme requires additional header to allow each relay to encode the received symbols transmitted to the next relay. We nevertheless show that the overhead due to the header vanishes as the field size of the code increases. We further derive an upper bound on the capacity in the proposed multi-hop relay network setup and show that in the special case when the link between the source and the first relay is subjected to the highest number of erasures, the upper bound is achieved up to an additional overhead (a required header).[1]

The rest of this paper is organized as follows. Section I-A outlines the network model of interest. Section I-B presents the formulation of streaming codes and outlines the known results for basic network models. Section I-C describes the results derived for the three-node network in [1] and discusses the issue with the upper bound [17]. Section I-D presents the main results of this paper. Section I-E we provide a motivating example for our proposed approach and demonstrate the sub-optimality of a straightforward extension of the state-independent coding scheme. Section II provides an upper bound for our proposed setting. Section III analyzes state-dependent SWDF coding scheme for a multi-hop relay network. We further analyze the size of the header required by the suggested scheme and discuss how the overhead associated with the header can be reduced in practice by either increasing the field size or by concatenating multiple copies of the code. Section IV provides an upper bound on the error probability of using the state-dependent SWDF coding scheme when used over a channel with random (i.i.d.) erasures. Section V provides numerical results for different coding schemes used over a four-node (two relay) network with random (i.i.d.) erasures. Finally, Section VI provides an extension of the presented results to the sliding window channel.

[1]Unfortunately, the proposed upper bound in [1] is incorrect in general, as we discuss in detail in this paper in Section I-C.

A. Network Model

A source node wants to send a sequence of messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ to a destination node with the help of L middle nodes $r_1, \ldots, r_L$. To ease notation, we denote the source node as r_0, and destination node as r_{L+1}. Let k be a non-negative integer, and $n_1, n_2, \ldots, n_{L+1}$ be $L+1$ natural numbers.

Each $\mathbf{s}_i$ is an element in $\mathbb{F}^k$ where $\mathbb{F}$ is some finite field. In each time slot $i \in \mathbb{Z}_+$, the source message $\mathbf{s}_i$ is encoded into a length-n_1 packet $\mathbf{x}_i^{(r_0)} \in \mathbb{F}^{n_1}$ to be transmitted to the first relay through the erasure channel (r_0, r_1). The relay receives $\mathbf{y}_i^{(r_1)} \in \mathbb{F}^{n_1} \cup \{*\}$ where $\mathbf{y}_i^{(r_1)}$ equals either $\mathbf{x}_i^{(r_0)}$ or the erasure symbol "$*$". In the same time slot, relay r_1 transmits $\mathbf{x}_i^{(r_1)} \in \mathbb{F}^{n_2}$ to relay r_2 through the erasure channel (r_1, r_2). Relay r_2 receives $\mathbf{y}_i^{(r_2)} \in \mathbb{F}^{n_2} \cup \{*\}$ where $\mathbf{y}_i^{(r_2)}$ equals either $\mathbf{x}_i^{(r_1)}$ or the erasure symbol "$*$". The same process continues (in the same time slot) until relay r_L transmits $\mathbf{x}_i^{(r_L)} \in \mathbb{F}^{n_{L+1}}$ to the destination r_{L+1} through the erasure channel (r_L, r_{L+1}). To simplify the analysis, we note that we assume zero propagation delay and zero processing delay for the transmission. Hence, in case no coding is applied ($n_1 = n_2 = \ldots = n_{L+1} = k$) and no erasures occur, $\mathbf{y}_i^{(r_{L+1})} = s_i$. When such assumptions are relaxed, extensions to the results described in the paper can be naturally described (see, e.g. [18]).

We first assume that on the discrete timeline, each channel (r_{j-1}, r_j) introduces up to N_j arbitrary erasures, respectively. The symbols generated in the L-node relay network at time i are illustrated in Figure 1.

B. Standard Definitions and Known Results

Definition 1: An $(n_1, n_2, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code consists of the following:

1) A sequence of source messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ where $\mathbf{s}_i \in \mathbb{F}^k$.
2) An encoding function $f_i^{(r_0)} : \underbrace{\mathbb{F}^k \times \ldots \times \mathbb{F}^k}_{i+1 \text{ times}} \rightarrow \mathbb{F}^{n_1}$ for each $i \in \mathbb{Z}_+$, where $f_i^{(r_0)}$ is used by node r_0 at time i to encode $\mathbf{s}_i$ according to
$$\mathbf{x}_i^{(r_0)} = f_i^{(r_0)}(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_i).$$
3) A relaying function for node r_j where $j \in \{1, \ldots, L\}$, $f_i^{(r_j)} : \underbrace{\mathbb{F}^{n_j} \cup \{*\} \times \ldots \times \mathbb{F}^{n_j} \cup \{*\}}_{i+1 \text{ times}} \rightarrow \mathbb{F}^{n_{j+1}}$ for each $i \in \mathbb{Z}_+$, where $f_i^{(r_j)}$ is used by node r_j at time i to construct
$$\mathbf{x}_i^{(r_j)} = f_i^{(r_j)}\left(\mathbf{y}_0^{(r_j)}, \mathbf{y}_1^{(r_j)}, \ldots, \mathbf{y}_i^{(r_j)}\right).$$

4) A decoding function ϕ_{i+T} : $\underbrace{\mathbb{F}^{n_{L+1}} \cup \{*\} \times \ldots \times \mathbb{F}^{n_{L+1}} \cup \{*\}}_{i+T+1 \text{ times}} \rightarrow \mathbb{F}^k$ for each $i \in \mathbb{Z}_+$ is used by node r_{L+1} at time $i+T$ to estimate $\mathbf{s}_i$ according to

$$\hat{\mathbf{s}}_i = \phi_{i+T}\left(\mathbf{y}_0^{(r_{L+1})}, \mathbf{y}_1^{(r_{L+1})}, \ldots, \mathbf{y}_{i+T}^{(r_{L+1})}\right). \quad (1)$$

Definition 2: An erasure sequence is a binary sequence denoted by $e^\infty \triangleq \{e_i\}_{i=0}^\infty$ where $e_i = \mathbf{1}\{\text{erasure occurs at time i}\}$.

An N-erasure sequence is an erasure sequence e^∞ that satisfies $\sum_{l=0}^\infty e_l = N$. Alternatively, we denote it as a N-deterministic erasure channel. The set of N-erasure sequences is denoted by Ω_N. We further denote $N_1, \ldots, N_{L+1}$ deterministic erasure network model as $N_1, \ldots, N_{L+1}$-erasure sequences, each occur on a different channel, where for any $j \in \{1, \ldots, L+1\}$, the maximal number of erasures on channel (r_{j-1}, r_j) is N_j.

Definition 3: The mapping $g_n : \mathbb{F}^n \times \{0,1\} \rightarrow \mathbb{F}^n \cup \{*\}$ of an erasure channel is defined as

$$g_n(\mathbf{x}, e) = \begin{cases} \mathbf{x} & \text{if } e = 0, \\ * & \text{if } e = 1. \end{cases} \quad (2)$$

Denoting with $e^{j+1} \in \Omega_{N_{j+1}}$ an admissible erasure sequence in channel (r_j, r_{j+1}), for any erasure sequence $e^{j,\infty}$ and any $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code, the following input-output relations holds for the erasure channel (r_j, r_{j+1}) for each $i \in \mathbb{Z}_+$:

$$\mathbf{y}_i^{(r_{j+1})} = g_{n_j}\left(\mathbf{x}_i^{(r_j)}, e_i^{j+1}\right)$$

Definition 4: An $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code is said to be $(N_1, N_2, \ldots, N_{L+1})$-achievable if the following holds for any N_j-erasure sequence $e^{j,\infty} \in \Omega_{N_j}$ ($j \in \{1, \ldots, L+1\}$), for all $i \in \mathbb{Z}_+$ and all $\mathbf{s}_i \in \mathbb{F}^k$, we have

$$\hat{\mathbf{s}}_i = \mathbf{s}_i$$

where

$$\hat{\mathbf{s}}_i = \phi_{i+T}\left(g_{n_{L+1}}\left(\mathbf{x}_0^{(r_L)}, e_0^{L+1}\right), \ldots, g_{n_{L+1}}\left(\mathbf{x}_{i+T}^{(r_L)}, e_{i+T}^{L+1}\right)\right) \quad (3)$$

and for previous nodes

$$\mathbf{x}_i^{(r_j)} = f_i^{(r_j)}\left(g_{n_j}\left(\mathbf{x}_0^{(r_{j-1})}, e_0^j\right), \ldots, g_{n_j}\left(\mathbf{x}_i^{(r_{j-1})}, e_i^j\right)\right). \quad (4)$$

Definition 5: The rate of an $(n_1, n_2, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code is $\frac{k}{n}$ where $n = \max\{n_1, n_2, \ldots, n_{L+1}\}$.

Definition 6: The $(T, N_1, \ldots, N_{L+1})$ capacity, denoted by $C_{T,N_1,\ldots,N_{L+1}}$, is the supremum rate achievable by $(n_1, \ldots, n_{j+1}, k, T)_{\mathbb{F}}$ streaming code that is $(N_1, \ldots, N_{L+1})$-achievable.

Remark 1: Throughout this work, we assume that n, k and the field size ($|\mathbb{F}|$) can be (individually) controlled by the system designer. Note that we can represent each symbol in $\mathbb{F}$ using $\log|\mathbb{F}|$ bits. Thus, the length of each channel packet is $n\log|\mathbb{F}|$ bits. In some applications the length of each packet may need to be fixed, although n and $\mathbb{F}$ may still be under the control of the system designer.

If, for a specific j, $N_l = 0$ for all $l \neq j$ and $N_j \neq 0$, then the L-node relay network with erasures reduces to a point-to-point packet erasure channel. It was previously shown in [10] that the maximum achievable rate of the point-to-point packet erasure channel with $N_j = N$ arbitrary erasures and delay of T symbols denoted by $C_{T,N}$ satisfies

$$C_{T,N} = \begin{cases} \frac{T-N+1}{T+1} & T \geq N \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

It was further shown that the capacity of the point-to-point channel with N arbitrary erasures and delay of T could be achieved by diagonally interleaving $(T+1, T-N+1)$ MDS code. We recall that for any natural numbers L and M, a systematic maximum-distance separable (MDS) $(L+M, L)$-code is characterized by an $L \times M$ parity matrix $\mathbf{V}^{L\times M}$ where any L columns of $\left[\mathbf{I}_L \ \mathbf{V}^{L\times M}\right] \in \mathbb{F}^{L\times(L+M)}$ are independent. It is well known that a systematic MDS $(L+M, L)$-code always exists as long as $|\mathbb{F}| \geq L+M$ [19]. To simplify notation, we sometimes denote $N_a^b = \sum_{l=a}^b N_l$. We will take all logarithms to base 2 throughout this paper. We denote the i'th element of vector $\mathbf{x}$ as $x[i]$, or sometime as $[x][i]$.

C. Known Results for Three Node Relay Network

In [1], a three node relay network (i.e., $L = 1$) was analyzed. We first discuss the upper bound stated in [1], and note an issue with the upper bound [17]. Assume a delay constraint of T, maximum of N_1 erasures in the source-relay link and maximum of N_2 erasures in the relay-destination link. The upper bound derived in [1] is, in fact, a minimization of two expressions $R \leq \min(R_1^+, R_2^+)$, where

$$R_1^+ = \frac{T-N_1-N_2+1}{T-N_2+1}, \quad R_2^+ = \frac{T-N_1-N_2+1}{T-N_1+1}.$$

We explain how the upper bound R_1^+ remains valid, but the upper bound R_2^+ is not valid [17].

Validity of R_1^+: Consider a cut-set bound where the relay and destination terminal cooperate, and there are N_1 erasures on the source-relay link. Following (5), a simple upper bound is:

$$R \leq \frac{T-N_1+1}{T+1}.$$

It is observed in [1] that one can tighten this bound by effectively reducing the delay from T to $T-N_2$. This can be justified as from the point of view of each source packet s_i, we can assume that the last N_2 packets in *each* interval $[i, i+T]$ for all $i \geq 0$, are erased on the relay-destination link. Although such an erasure pattern is clearly worse than our assumed model (as we assumed a global limit of N_2 erasures on the relay-destination link), it can be shown that the decoder must be able to recover every packet $\mathbf{s}_i$ for such a pattern. In particular, since the source does not have any information with respect to the erasures (on any of the links) it must handle the worst-case scenario (i.e., when a burst of N_2 erasures happens in the last N_2 positions in the interval $[i, i+T]$ for the recovery of source packet $\mathbf{s}_i$) for each i). This leads to the following upper bound.

Theorem 1 ((Eq. (18) in [1]): For any (T, N_1, N_2), recalling that the point-to-point capacity satisfies (5), we have

$$R \leq C_{T-N_2,N_1} = \begin{cases} \frac{T-N_1-N_2+1}{T-N_2+1} & T \geq N_1 + N_2 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Remark 2: Our upper bound presented for $L + 1$ node relay network in Section II is, in fact, a generalization of Theorem 1 (when $L = 1$, the proof presented in Section II proves Theorem 1).

Validity of R_2^+: In [1], it was further argued that we can upper bound $R \leq R_2^+$ as well. In order to argue this, a cut-set bound where the source and relay cooperate was considered when there are N_2 erasures on the relay-destination link. The naive upper bound again will be

$$R \leq \frac{T - N_2 + 1}{T + 1}.$$

The authors then attempted to argue that the delay can be further reduced from T to $T - N_1$. To claim this, the authors assumed that in each interval $[i, i+T]$, if the first N_1 packets are erased, the decoder should still be able to recover all the source packets. Unfortunately, this argument does not hold, as the relay node is aware of the erasures on the source-relay link. Thus, a scheme that is designed for a global constraint of N_1 erasures on the source-relay link may not be applicable to the case when we introduce a more stringent erasure pattern. In fact, in [20], similar ideas have been utilized to improve the achievable rate over the rate in [1] and the present paper for the case when $L = 1$.

As an achievable scheme, a coding scheme coined SWDF was presented in [1] and shown to achieve:

$$R = \frac{T - N_1 - N_2 + 1}{T - \min(N_1, N_2) + 1}. \tag{7}$$

In this paper we will refer to this scheme as state-independent SWDF for reasons that will be explained later. Although [1] incorrectly claims this to be the capacity for all N_1 and N_2, we note that the expression in (7) achieves the upper bound of Theorem 1 (and thus establishes capacity) when $N_1 \geq N_2$.

D. Main Results

In this paper, we first derive a simple upper bound on the achievable rate in $L+1$-node relay network as a generalization of Theorem 1.

Theorem 2: Assume a network with L relays. For a target overall delay of T, where the maximal number of arbitrary erasures in link (r_{j-1}, r_j), $j \in \{1, \ldots, L+1\}$ is N_j. The capacity is upper bounded by

$$R \leq \begin{cases} \frac{T-\sum_{l=1}^{L+1} N_l+1}{T-\sum_{l=2}^{L+1} N_l+1} = C_{T-\sum_{l=2}^{L+1} N_l, N_1} & T \geq \sum_{l=1}^{L+1} N_l \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

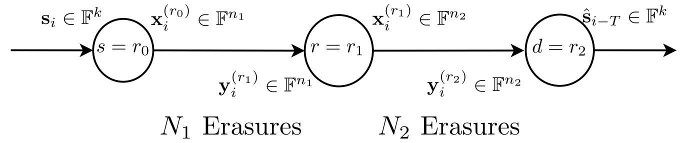

Fig. 2. A three-node relay network.

Denoting

$$n_{\max} \triangleq \max_{j \in \{1,\ldots,L+1\}} \left(T - \sum_{l=1, l \neq j}^{L+1} N_l + 1 \right), \tag{9}$$

we show the following rate is achievable using a new coding scheme that we call state-dependent SWDF.

Theorem 3: Assume a link with L relays. For a target overall delay of T, where the maximal number of arbitrary erasures in link (r_j, r_{j+1}), $j \in \{0, \ldots, L\}$ is N_{j+1} and $T \geq \sum_{l=1}^{L+1} N_l$. When $|\mathbb{F}| \geq n_{\max}$, the following rate is achievable.

$$R \geq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \left(\sum_{l=1, l \neq j}^{L+1} N_l \right) + 1 + \frac{n_{\max} \lceil \log(n_{\max}) \rceil}{\log(|\mathbb{F}|)}} \tag{10}$$

where $n_{\max}$ is defined in (9).

Remark 3: When $N_1 \geq N_j$, $\forall j \neq 1$, and $|\mathbb{F}| \to \infty$, the achievable rate of state-dependent SWDF (10) achieves the upper bound of Theorem 2 (and thus establishes capacity).

Remark 4: Comparing (7) and (10) we note that when $|\mathbb{F}| \to \infty$ state-dependent SWDF used over three node network achieves the same rate as state-independent SWDF.

Remark 5: Although the deterministic erasure model is formulated in such a way that link (r_{j-1}, r_j) introduces only a finite number of erasures over the discrete timeline, the suggested upper bound and the suggested achievable coding rate remains unchanged for the following sliding window model that can introduce infinitely many erasures on each communication link. Every message must be perfectly recovered with delay T as long as the number of erasures introduced in link (r_{j-1}, r_j) in any sliding window of size T + 1 does not exceed N_j. This is further described in Section VI.

E. Motivating Example

Consider a link with up to $N = 2$ arbitrary erasures, where the delay constraint the transmission has to meet is $T = 3$ packets. The capacity of this link according to (5) is $C_{3,2} = 2/4$. Now, assume that in fact, this link is a three-node network ($L = 1$), where up to $N_1 = 1$ erasures occur in link (r_0, r_1) and up to $N_2 = 1$ erasures occur in link (r_1, r_2), where transmission has to be decoded with the same overall delay of $T = 3$ packets. This network is depicted in Figure 2. The maximal achievable rate of this network according to (6) is $2/3$, which is better than the point-to-point link.

We start by recalling the coding scheme that was shown in [1] to achieve this upper bound. The example we show next is the same as provided in [1] for the scenario described above

TABLE I

A STATE-LESS SWDF STRATEGY FOR A SINGLE RELAY. SYMBOLS BELONG TO THE SAME CODE USED BY THE SOURCE (s) AND SYMBOLS BELONG TO THE SAME CODE USED BY THE RELAY (r) ARE MARKED WITH A FRAME WITH THE SAME STYLE

Time: i	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$	$i+5$
a_i	a_{i-1}	a_i	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}	a_{i+5}
b_i	b_{i-1}	b_i	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}	b_{i+5}
$a_{i-2}+b_{i-1}$	$a_{i-3}+b_{i-2}$	$a_{i-2}+b_{i-1}$	$a_{i-1}+b_i$	a_i+b_{i+1}	$a_{i+1}+b_{i+2}$	$a_{i+2}+b_{i+3}$	$a_{i+3}+b_{i+4}$

(a) Symbols transmitted by the source node s.

Time: i	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$	$i+5$
b_{i-1}	b_{i-2}	b_{i-1}	b_i	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}
a_{i-2}	a_{i-3}	a_{i-2}	a_{i-1}	a_i	a_{i+1}	a_{i+2}	a_{i+3}
$a_{i-3}+b_{i-3}$	$a_{i-4}+b_{i-4}$	$a_{i-3}+b_{i-3}$	$a_{i-2}+b_{i-2}$	$a_{i-1}+b_{i-1}$	a_i+b_i	$a_{i+1}+b_{i+1}$	$a_{i+2}+b_{i+3}$

(b) Symbols transmitted by relay node r.

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$	$i+5$
a_{i-3}	a_{i-4}	a_{i-3}	a_{i-2}	a_{i-1}	a_i	a_{i+1}	a_{i+2}
b_{i-3}	b_{i-4}	b_{i-3}	b_{i-2}	b_{i-1}	b_i	b_{i+1}	b_{i+2}

(c) Estimates constructed by the destination node d.

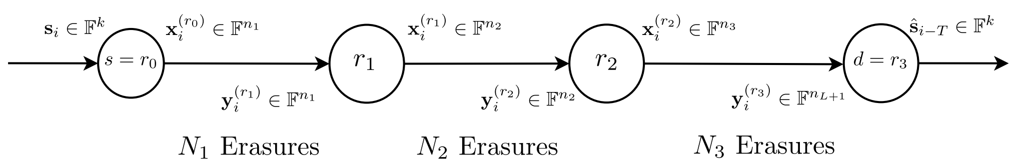

Fig. 3. A four-node relay network.

TABLE II

SYMBOLS TRANSMITTED BY NODE r_2 WHEN TRYING TO EXTEND THE STATE-LESS SWDF STRATEGY. SYMBOLS BELONG TO THE SAME CODE ARE DENOTED WITH FRAME WITH THE SAME STYLE

Time: i	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$	$i+5$
a_{i-3}	a_{i-4}	a_{i-3}	a_{i-2}	a_{i-1}	a_i	a_{i+1}	a_{i+2}
b_{i-3}	b_{i-4}	b_{i-3}	b_{i-2}	b_{i-1}	b_i	b_{i+1}	b_{i+2}
$a_{i-5}+b_{i-4}$	$a_{i-6}+b_{i-5}$	$a_{i-5}+b_{i-4}$	$a_{i-4}+b_{i-3}$	$a_{i-3}+b_{i-2}$	$a_{i-2}+b_{i-1}$	$a_{i-1}+b_i$	a_i+b_{i+1}

(in which the total delay $T = 3$ and the maximal number of erasures in each channel is $N_1 = N_2 = 1$).

Suppose node s transmits two bits a_i and b_i at each discrete time $i \geq 0$ to node d. For each time i, node s transmits the three-symbol packet $\mathbf{x}_i^{(r_0)} = [a_i \;\; b_i \;\; a_{i-2}+b_{i-1}]$ according to Table I where $a_j = b_j = 0$ for any $j < 0$ by convention, and the symbols highlighted in the same color are generated by the same block code.

Since channel (s, r) introduces at most $N_1 = 1$ erasure, each a_i and each b_i can be perfectly recovered by node r by time $i+2$ and time $i+1$ respectively. Therefore, at each time i, node r should have recovered a_{i-2} and b_{i-1} perfectly with delays 2 and 1 respectively, and it will re-encode them into another three-symbol packet $\mathbf{x}_{i+1}^{(r_1)} = [b_{i-1} \;\; a_{i-2} \;\; b_{i-3}+a_{i-3}]$. This is depicted in Table I. Since b_{i-3}, a_{i-3} and $b_{i-3}+a_{i-3}$ are transmitted by node r at time $i-2$, $i-1$ and i respectively, it follows from assuming that channel (r, d) introduces at most $N_2 = 1$ erasure, that node d can recover a_{i-3} and b_{i-3} by time i. Consequently, this SWDF strategy achieves a rate of 2/3.

An important feature of this code is the fact that it is a state-less code, i.e., the structure of the code does not depend on the specific erasure pattern at any of the segments. However, if another relay is to be considered (i.e., the destination is now replaced with relay r_2), assuming up to $N_3 = 1$ erasure in link (r_2, d), if we directly concatenate the state-independent SWDF schemes in a simple way, we can support the same rate ($R = 2/3$) only if we increase the delay requirement to $T = 5$. The network of interest is depicted in Figure 3.

This can be seen since basically using the coding scheme in Table I, relay r_2 can be viewed as source s with delay of $T = 3$ packets (essentially, the delay of different symbols is "reset"). Due to causality, relay r_2 can only use the coding scheme of the sender, depicted in Table II. It can be viewed

TABLE III

TRANSMISSION OF THE SOURCE IN CHANNEL (r_0, r_1). SYMBOLS BELONG TO THE SAME CODE ARE MARKED WITH THE SAME COLOR

Time: i	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header: 123	123	123	123	123	123	123
a_i	a_{i-1}	a_i	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}
b_i	b_{i-1}	b_i	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}
$a_{i-2}+$ b_{i-1}	$a_{i-3}+$ b_{i-2}	$a_{i-2}+$ b_{i-1}	$a_{i-1}+$ b_i	a_i+ b_{i+1}	$a_{i+1}+$ b_{i+2}	$a_{i+2}+$ b_{i+3}

TABLE IV

TRANSMISSION OF RELAY r_1 IN CHANNEL (r_1, r_2), GIVEN THAT SYMBOL i WAS ERASED WHEN TRANSMITTED IN LINK (r_0, r_1). SYMBOLS BELONG TO THE SAME CODE ARE MARKED WITH THE SAME COLOR

Time: $i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header: 123	123	223	113	123	123
a_{i-2}	a_{i-1}	b_{i+1}	a_{i+1}	a_{i+2}	a_{i+3}
b_{i-2}	b_{i-1}	b_i	a_i	b_{i+2}	b_{i+3}
$a_{i-4}+$ b_{i-3}	$a_{i-3}+$ b_{i-2}	$a_{i-2}+$ b_{i-1}	$a_{i-1}+$ b_i	a_i+ b_{i+1}	$a_{i+1}+$ b_{i+2}

that when $N_3 = 1$, symbol a_i is guaranteed to be recovered only at time $i + 5$. To the best of our knowledge, there is no extension of state-independent SWDF which results in lower delay.

In this paper, we suggest a state-dependent scheme, i.e., a scheme in which the order of symbols in each block code used (and thus the order in each diagonal) is set according to the erasure pattern in the previous link. As we demonstrate later, the order of the symbols is transmitted to the receiver to allow decoding. Hence, additional overhead is required. We first show an example of the suggested scheme to the three-node network, and then show how to extend it to a four-node network.

In the proposed scheme, the source r_0 uses the same code suggested in [1], i.e., a $(3, 2)$ MDS code that can recover any arbitrary single erasure with a delay of two symbols combined with diagonal interleaving (i.e., the block code is applied on the diagonals). For each time i, the node s transmits the three-symbol packet $\mathbf{x}_i^{(r_0)} = [a_i, b_i, a_{i-2} + b_{i-1}]$.

When there are no erasures, relay r_1 uses the same code as the source r_0 *while delaying it by one symbol*, i.e. relay r_1 sends the following three-symbol packet $\mathbf{x}_{i+1}^{(r_1)} = [a_i, b_i, a_{i-2} + b_{i-1}]$. If an erasure occurred, the relay would send any available symbols (per diagonal) in the order they were received until it can decode the information symbols from this block code. Then, the erased symbols will be sent. For example, assume that the packet sent at time i from the source was erased in link (r_0, r_1). Relay r_1 will send $\mathbf{x}_{i+1}^{(r_1)} = [b_{i+1}, b_i, a_{i-2} + b_{i-1}]$ and $\mathbf{x}_{i+2}^{(r_1)} = [a_{i+1}, a_i, a_{i-1} + b_i]$ as depicted in Table IV below. Again, symbols belong to the same block code are marked with the same color. Further, headers which are different than the ones used by r_0 are marked with a frame.

We note that the erasure in time i in link (r_0, r_1) caused a change in the order of the symbols in packets $i + 1$ and $i+2$. Denoting the order of symbols sent from r_0 in each code (alternatively on each diagonal) as $[1, 2, 3]$, in this example, the header of each packet is composed of the location of the symbols from each block code (the order of the symbols in each diagonal) as they would appear in the code used by r_0.

We show next that the MDS code used by each relay on each diagonal can be viewed as using a punctured version of the MDS code that is used by the "bottleneck" relay, which is the lowest rate MDS code being used. Hence, we note that it can be viewed as if the suggested coding scheme only changes the order of symbols per diagonal (taken from the MDS code with the lowest rate), and it does not add or remove symbols. Therefore, using a single index per symbol suffices to allow recovery at each destination.

In our example, as can bee seen in Table IV, at time i relay r_1 sends $\mathbf{x}_i^{(r_1)} = [a_{i-1} \;\; b_{i-1} \;\; a_{i-3} + b_{i-2}]$ with header "123" indicating that the first symbol is the first symbol (marked with underline) in the code $[\underline{a_{i-1}} \;\; b_i \;\; a_{i-1} + b_i]$, the second symbol is the second symbol in the code $[a_{i-2} \;\; \underline{b_{i-1}} \;\; a_{i-2} + b_{i-1}]$ and the third symbol is the code $[a_{i-3} \;\; b_{i-2} \;\; \underline{a_{i-3} + b_{i-2}}]$.

However, following the erasure occurred at link (r_0, r_1) at time i, relay r_1 can not send symbol a_i at time $i+1$ as planned (as if there was no erasure). However, it can send b_{i+1} which was not erased. With respect to the second symbol, we note that since a_{i-1} and $a_{i-1}+b_i$ were received, b_i can be recovered and used. Similarly, since a_{i-2} and b_{i-1} were received, symbol $a_{i-2}+b_{i-1}$ can be generated and used. Thus, at time $i+1$ the relay can send $\mathbf{x}_{i+1}^{(r_1)} = [b_{i+1} \;\; b_i \;\; a_{i-2} + b_{i-1}]$. To indicate the change in order of the symbols used in the first code, the header is changed to "223" which indicates that the first symbol is the second symbol in the code associated with this diagonal.

At time $i + 2$, the relay can recover and send a_i. Hence, it sends $\mathbf{x}_{i+2}^{(r_1)} = [a_{i+1} \;\; a_i \;\; a_{i-1} + b_i]$ with header "113" indicating now that the second symbol is the first symbol in the code associated with this diagonal. It can be easily verified that the destination can recover the original data at a delay of $T = 3$ (assuming any single arbitrary erasure in the link between the relay and destination).

This concept can be applied to additional relays if they exist. For example consider four-node network $(L = 2)$. The transmission scheme on the next relay r_2 (in this specific example) is the same as the transmission scheme of the first relay, i.e., in case there is no erasure, transmit (on each diagonal)

TABLE V

TRANSMISSION OF RELAY r_2 IN CHANNEL (r_2, r_3), GIVEN THAT SYMBOL $i+2$ WAS ERASED WHEN TRANSMITTED IN LINK (r_1, r_2). SYMBOLS BELONG TO THE SAME CODE ARE MARKED WITH THE SAME COLOR

Time: $i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header: 123	123	123	223	213	113
a_{i-3}	a_{i-2}	a_{i-1}	b_{i+1}	b_{i+2}	a_{i+2}
b_{i-3}	b_{i-2}	b_{i-1}	b_i	a_i	a_{i+1}
$a_{i-5}+ b_{i-4}$	$a_{i-4}+ b_{i-3}$	$a_{i-3}+ b_{i-2}$	$a_{i-2}+ b_{i-1}$	$a_{i-1} b_i$	$a_i+ b_{i+1}$

the symbols in the same order as received, delayed by one symbol (i.e., a total delay of two symbols from the sender). If an erasure occurred before r_2 could decode the information symbols (per diagonal), the relay will forward the non-erased symbols (as we show next, it is guaranteed that there will be enough such symbols until the relay could decode the information symbols). When the information symbols can be decoded, the relay will transmit the erased symbols.

For example, in case the symbol transmitted from relay r_1 to r_2 at time $i+2$ is erased, the suggested transmission scheme of relay r_2 is given in Table V below. Again, symbols that belong to the same block code are marked with the same color. Further, headers that are different than the ones used by relay r_1 are marked with a frame.

We note that the erasure in time $i+2$ in link (r_1, r_2) caused a change in order of the symbols in packets $i+3$ and $i+4$ (the order of symbols in time $i+2$ is not the original order, yet it is the same as was transmitted from r_1 at time $i+1$). Since the same $(3,2)$ code is used (with a different order of symbols which is communicated to the receiver), it can be easily verified that each packet can be recovered up to delay of $T=4$ symbols for any arbitrary erasure in the link between the relay and the destination.

In this example, the maximal size of the header is three numbers, each taken from $\{1,2,3\}$. Hence, its maximal size of the header is $3\log(3)$ bits. Since the block code used in each link transmits two bits using three bits in every channel use, we conclude that the scheme achieves a rate of

$$R = \frac{2}{3+3\lceil\log(3)\rceil}. \tag{11}$$

We show next that this idea can be extended to transmitting symbols taken from any field $\mathbb{F}$ thus, in general, the achievable rate when $T=4$ and $N_1=N_2=N_3=1$ is

$$\begin{aligned} R &= \frac{2\cdot\log(|\mathbb{F}|)}{3\cdot\log(|\mathbb{F}|)+3\lceil\log(3)\rceil} \\ &= \frac{2}{3+\frac{3\lceil\log(3)\rceil}{\log(|\mathbb{F}|)}}, \end{aligned} \tag{12}$$

which approaches $2/3$ as the field size increases. As we further show next, the upper bound for this scenario is indeed $2/3$.

II. PROOF OF THE UPPER BOUND

Fix any $(N_1,\dots,N_{L+1},T)$. Suppose we are given an $(N_1,\dots,N_{L+1})$-achievable $(n_1,\dots,n_{j+1},k,T)_{\mathbb{F}}$-streaming code for some $n_1,\dots,n_{j+1},k$ and $\mathbb{F}$. Our goal is to show that

$$\frac{k}{\max\{n_1,\dots,n_{L+1}\}} \le \frac{T-\sum_{l=1}^{L+1}N_l+1}{T-\sum_{l=2}^{L+1}N_l+1}. \tag{13}$$

when $T \ge \sum_{l=1}^{L+1}N_l$ and equals zero otherwise.

To this end, we let $\{\mathbf{s}_i\}_{i\in\mathbb{Z}_+}$ be i.i.d. random variables where $\mathbf{s}_0$ is uniform on $\mathbb{F}^k$. Since the $(n_1,\dots,n_{j+1},k,T)_{\mathbb{F}}$-streaming code is $(N_1,\dots,N_{L+1})$-achievable, it follows from Definition 4 that

$$H\left(\mathbf{s}_i \,\middle|\, \mathbf{y}_0^{r_{L+1}}, \mathbf{y}_1^{r_{L+1}}, \dots, \mathbf{y}_{i+T}^{r_{L+1}}\right) = 0 \tag{14}$$

for any $i\in\mathbb{Z}_+$ and any $e^{j,\infty}\in\Omega_{N_j}$. Consider the two cases.

Case $T<\sum_{l=1}^{L+1}N_l$:

Let $e^{j,\infty}\in\Omega_{N_j}$ be the error sequence on link (r_{j-1},r_j) where $j\in\{1,2,\dots,L+1\}$ such that

$$e_i^{j,\infty} = \begin{cases} 1 & \text{if } \sum_{l=1}^{j-1}N_l \le i \le \sum_{l=1}^{j}N_l - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

We note that (15) means that $\mathbf{y}_{N_1}^{(r_1)}$ is the first packet which can be used to recover $\mathbf{s}_0$ at r_1. Further, $\mathbf{y}_{N_1+N_2}^{(r_2)}$ is the first packet which can be used to recover $\mathbf{s}_0$ at r_2. Continuing the transmission across all other relays, it follows that $\mathbf{y}_{\sum_l N_l}^{(r_{L+1})}$ is the first packet which can be used to recover $\mathbf{s}_0$. Since $T<\sum_{l=1}^{L+1}N_l$ it follows the delay constraint can not be met.

Hence, due to (15) and Definition 1, we have

$$I\left(\mathbf{s}_0; \mathbf{y}_0^{r_{L+1}}, \mathbf{y}_1^{r_{L+1}}, \dots, \mathbf{y}_T^{r_{L+1}}\right) = 0. \tag{16}$$

Combining (14), (16) and the assumption that $T < \sum_{l=1}^{L+1}N_l$, we obtain $H(\mathbf{s}_0)=0$. Since $\mathbf{s}_0$ consists of k uniform random variables in $\mathbb{F}$, the only possible value of k is zero, which implies

$$\frac{k}{\max\{n_1,\dots,n_{L+1}\}} = 0. \tag{17}$$

Case $T\ge\sum_{l=1}^{L+1}N_l$:

The proof follows the footsteps of [1] for the link between the source and the first relay (link (r_0,r_1)). First we note that for every $i\in\mathbb{Z}_+$, message $\mathbf{s}_i$ has to be perfectly recovered by node r_1 by time $i+T-\sum_{l=2}^{L+1}N_l$ given that $\mathbf{s}_0,\mathbf{s}_1,\dots,\mathbf{s}_{i-1}$ have been correctly decoded by node r_1, or otherwise a length N_2 burst erasure from time $i+T-\sum_{l=2}^{L+1}N_l+1$ to $i+T-\sum_{l=3}^{L+1}N_l$ introduced on channel (r_1,r_2) followed by a length N_3 burst erasure from time $i+T-\sum_{l=3}^{L+1}N_l+1$ to $i+T-\sum_{l=4}^{L+1}N_l$ introduced on channel (r_2,r_3) and so on until a length N_{L+1} burst erasure from time $i+T-N_{L+1}+1$ to $i+T$ would result in a decoding failure for node r_1, node r_2 and all the nodes up to the destination r_{L+1}.[2]

[2] Although we didn't formally require the relays to decode the message, analyzing the suggested erasure pattern shows that all nodes have exactly the same time indices, $[i : i+T-N_2^{L+1}]$, available for processing $\mathbf{s}_i$. Recalling from Definition 1 that the relaying functions are casual, requiring that the destination can recover message $\mathbf{s}_i$ given that $\mathbf{s}_0,\mathbf{s}_1,\dots,\mathbf{s}_{i-1}$ have been correctly decoded by the destination from these packets is equivalent to requiring that each of the nodes can recover it from the same packets (given that all past messages were correctly recovered).

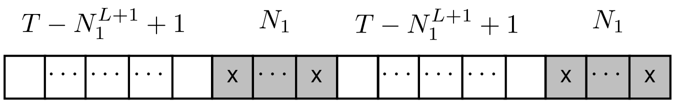

Fig. 4. A periodic erasure sequence with period $T - N_2^{L+1} + 1$.

Recalling that $N_2^{L+1} = \sum_{l=2}^{L+1} N_l$ it then follows that

$$H\left(\mathbf{s}_i \,\middle|\, \left\{\mathbf{x}_i^{(r_0)}, \mathbf{x}_{i+1}^{(r_0)}, \ldots, \mathbf{x}_{i+T-N_2^{L+1}}^{(r_0)}\right\} \setminus \left\{\mathbf{x}_{\theta_1}^{r_0}, \ldots, \mathbf{x}_{\theta_{N_1}}^{r_0}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{i-1}\right) = 0 \tag{18}$$

for any $i \in \mathbb{Z}_+$ and N_1 non-negative integers denoted by $\theta_1, \ldots, \theta_{N_1}$. We analyze the following periodic erasure pattern on the first link. Assume that the last N_1 packets in windows $[0 : 1 \cdot (T - N_2^{L+1} + 1) - 1], [T - N_2^{L+1} + 1 : 2 \cdot (T - N_2^{L+1} + 1) - 1], \ldots, [m \cdot (T - N_2^{L+1} + 1) : (m+1) \cdot (T - N_2^{L+1} + 1) - 1]$ for all $m \geq 0$ are erased.

Appendix A describes a specific erasure pattern for which for each $j \in \mathbb{N}$ we have

$$H\left(\mathbf{s}_0, \ldots, \mathbf{s}_{j\cdot(T-N_2^{L+1}+1)} \,\middle|\, \left\{\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)}^{(r_0)}, \mathbf{x}_{m\cdot(T-N_2^{L+1}+1)+1}^{(r_0)}, \ldots, \mathbf{x}_{m\cdot(T-N_2^{L+1}+1)+T-N_1-N_2^{L+1}}^{(r_0)}\right\}_{m=0}^{j}\right) = 0, \tag{19}$$

where the conditional entropy involves $j(T - N_2^{L+1} + 1) + 1$ source messages and $(j+1)(T - N_1^{L+1} + 1)$ channel packets. Further, it can be seen in Figure 9, in case a packet sent at time i is erased, it is recovered (at the latest) at time $i + T - N_2^{L+1} + 1$.

This conditional entropy is, in fact, the conditional entropy of a point-to-point streaming code with rate k/n_1 and delay $T - N_2^{L+1}$ which was designed to recover any N_1 erasures. In particular, the point-to-point code can recover from the periodic erasure sequence $\tilde{e}^\infty$ depicted in Figure 4, which is formally defined as

$$\tilde{e}_i = \begin{cases} 0 & \text{if } 0 \leq i \mod (T - N_2^{L+1} + 1) \leq T - N_1^{L+1} \\ 1 & \text{otherwise} \end{cases} \tag{20}$$

By standard arguments which are rigorously elaborated in Appendix B, we conclude that

$$\begin{aligned} \frac{k}{\max\{n_1, \ldots, n_{L+1}\}} &\leq \frac{k}{n_1} \\ &\leq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \sum_{l=2}^{L+1} N_l + 1} \\ &= C_{T - \sum_{l=2}^{L+1} N_l, N_1}. \end{aligned} \tag{21}$$

III. State-Dependent SWDF

As mentioned above, the achievable scheme we analyze is a state-dependent SWDF scheme. This scheme is composed of a block code combined with diagonal interleaving. We start with some basic definitions of point-to-point block codes, which would be the basis for this scheme.

Definition 7: A point-to-point $(n, k, T)_{\mathbb{F}}$-block code consists of the following

1) A sequence of k symbols $\{u[l]\}_{l=0}^{k-1}$ where $u[l] \in \mathbb{F}$.
2) A generator matrix $\mathbf{G} \in \mathbb{F}^{k \times n}$. The source codeword is generated according to

$$[x[0]\ x[1]\ \ldots\ x[n-1]] = [u[0]\ u[1]\ \ldots\ u[k-1]]\,\mathbf{G} \tag{22}$$

3) A decoding function $\varphi_{l+T} : \underbrace{\mathbb{F} \cup \{*\} \times \ldots \times \mathbb{F} \cup \{*\}}_{n \text{ times}} \to \mathbb{F}$ for each $l \in \{0, 1, \ldots, k-1\}$, where φ_{l+T} is used by the destination at time $\min(l+T, n-1)$ to estimate $u[l]$ according to

$$\hat{u}[l] = \begin{cases} \varphi_{l+T}(y[0], y[1], \ldots, y[l+T]) & \text{if } l + T \leq n - 1 \\ \varphi_{l+T}(y[0], y[1], \ldots, y[n-1]) & \text{if } l + T > n - 1 \end{cases} \tag{23}$$

Definition 8: A point-to-point $(n, k, T)_{\mathbb{F}}$-block code is said to be N-achievable if the following holds for any N-erasure sequence $e^\infty \in \Omega_N$: For the $(n, k, T)_{\mathbb{F}}$-block code, we have

$$\hat{u}[l] = u[l] \tag{24}$$

for all $l \in \{0, 1, \ldots, k-1\}$ and all $u[l] \in \mathbb{F}$, where

$$\hat{u}[l] = \begin{cases} \varphi_{l+T}(g_1(x[0], e_0), \ldots, g_1(x[l+T], e_{l+T})) & l + T \leq n - 1 \\ \varphi_{l+T}(g_1(x[0], e_0), \ldots, g_1(x[n-1], e_{n-1})) & l + T > n - 1 \end{cases} \tag{25}$$

with g_1 being the symbol-wise erasure function defined in (2).

We define

$$\begin{aligned} k &= T - \sum_{l=1}^{L+1} N_l + 1 \\ n_{j+1} &= T - \sum_{l=1, l \neq j+1}^{L+1} N_l + 1 \\ T_{j+1} &= T - \sum_{l=1, i \neq j+1}^{L+1} N_l. \end{aligned} \tag{26}$$

and show that while ignoring the additional header, the rate of the suggested coding scheme in channel (r_j, r_{j+1}) is

$$R_j = \frac{k}{n_{j+1}}.$$

Hence, the overall rate (again, ignoring the additional header) is

$$\begin{aligned} R &= \frac{k}{\max_j\{n_{j+1}\}} \\ &= \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \sum_{l=1, l \neq j}^{L+1} N_l + 1} \\ &\triangleq C^+_{T, N_1, \ldots, N_{L+1}} \end{aligned}$$

The suggested coding scheme is composed of $(n_{j+1}, k, N_1^j + T_j)_{\mathbb{F}}$-block codes combined with diagonal interleaving. Each relay r_j is using a collection of $(n_{j+1}, k, N_1^j + T_j)_{\mathbb{F}}$-block codes, where the specific code that depends on the erasure pattern in the previous relay, i.e., each code can be different.

TABLE VI
AN EXAMPLE OF A SINGLE CODE TRANSMITTED IN r_j

Time: $i+N_1^j$	$i+N_1^j+1$	$\cdots$	$i+N_1^j+n_{j+1}-1$
$[\tilde{\mathbf{s}}_i \times \mathbf{G}_i^{(r_j)}][0]$			
	$[\tilde{\mathbf{s}}_i \times \mathbf{G}_i^{(r_j)}][1]$		
		$\ddots$	
			$[\tilde{\mathbf{s}}_i \times \mathbf{G}_i^{(r_j)}][n_{j+1}-1]$

More formally, let $s_i[l]$ be the l'th symbol of the source message $\mathbf{s}_i$ and let $x_i^{(r_j)}[l]$ be the l'th symbol of the output of encoding function $f_i^{(r_j)}$ in relay r_j. For each $i \in \mathbb{Z}_+$, a single transmission function of relay r_j constructs

$$\left[x_{i+N_1^j}^{(r_j)}[0]\; x_{i+N_1^j+1}^{(r_j)}[1]\; \cdots\; x_{i+N_1^j+n_{j+1}-1}^{(r_j)}[n_{j+1}-1]\right] \triangleq [s_i[0]\; s_{i+1}[1]\; \cdots\; s_{i+k-1}[k-1]] \times \mathbf{G}_i^{(r_j)}, \tag{27}$$

where we show next that $\mathbf{G}_i^{(r_j)}$ is a $k \times n_{j+1}$ generator matrix of an (n_{j+1}, k) MDS code (not necessarily a systematic code as the order of transmission is a function of the erasure pattern in previous links). We assume that for any $i < 0$, $\mathbf{s}_i = \mathbf{0}$.

Denoting with

$$\tilde{\mathbf{s}}_i = [s_i[0]\; s_{i+1}[1]\; \cdots\; s_{i+k-1}[k-1]], \tag{28}$$

an example of the diagonal interleaving of a single code is given in Table VI below.

Therefore, each transmitted packet from relay r_j is composed of n_{j+1} symbols, each of which is taken from a different block code. Recalling that $[\tilde{\mathbf{s}}_i \times \mathbf{G}_i^{(r_j)}][j]$ means that we take the j'th element from the vector resulting from multiplying $\tilde{\mathbf{s}}_i$ with the generator matrix $\mathbf{G}_i^{(r_j)}$, an example of a packet sent by relay r_j is given in (29) below.

$$\mathbf{x}_{i+N_1^j}^{(r_j)} = \begin{bmatrix} x_{i+N_1^j}^{(r_j)}[0] \\ x_{i+N_1^j}^{(r_j)}[1] \\ \vdots \\ x_{i+N_1^j}^{(r_j)}[n_{j+1}-1] \end{bmatrix} = \begin{bmatrix} [\tilde{\mathbf{s}}_i \times \mathbf{G}_i^{(r_j)}][0] \\ [\tilde{\mathbf{s}}_{i-1} \times \mathbf{G}_{i-1}^{(r_j)}][1] \\ \vdots \\ [\tilde{\mathbf{s}}_{i-n_{j+1}+1} \times \mathbf{G}_{i-n_{j+1}+1}^{(r_j)}][n_{j+1}-1] \end{bmatrix} \tag{29}$$

We note again that the specific structure of each $\mathbf{G}_i^{(r_j)}$ is defined according to the erasure pattern of the previous links.

We describe next the process of generating $\mathbf{G}_i^{(r_j)}$.

- At the sender (r_0), use an (n_1, k) MDS code. Hence, $\mathbf{G}_i^{(r_0)}$ is the generator matrix of an (n_1, k) MDS code.
- Each encoder at relay r_j ($j \in \{1, \ldots, L\}$) performs the following
  1) Until time $i + N_1^j - 1$, store any non-erased symbols from the first N_j received symbols from link (r_{j-1}, r_j), i.e., all non-erased symbols from
  $$\left\{x_{i+N_1{}^{j-1}}^{(r_{j-1})}[0], \ldots, x_{i+N_1{}^{j-1}+N_j-1}^{(r_{j-1})}[N_j-1]\right\}. \tag{30}$$
  2) Start transmitting at time $i + N_1^j$ (while continuing to store the received symbols from link (r_{j-1}, r_j)). Until time $i+N_1^j+k-2$, forward the $k-1$ symbols received from link (r_{j-1}, r_j) by the order they were received. Noting that $N_j + k - 1 = n_j - 1$, relay r_j forward the $k-1$ non-erased symbols from
  $$\left\{x_{i+N_1{}^{j-1}}^{(r_{j-1})}[0], \ldots, x_{i+N_1{}^{j-1}+n_j-2}^{(r_{j-1})}[n_j-2]\right\}. \tag{31}$$
  3) At time $i + N_1^j + k - 1$, recover $\tilde{\mathbf{s}}_i$ (assuming that the code used over link (r_{j-1}, r_j) is an (n_j, k) MDS code, relay r_j has now k symbols from which $\tilde{\mathbf{s}}_i$ can be recovered). In Lemma 1 below we prove that it is feasible for any $N_1, \ldots, N_{L+1}$-erasure sequence.
  4) Transmit until time $i + N_1^j + n_{j+1} - 1$ re-encoded symbols.[3] The encoded symbols should be non-received symbols from an (n_{max}, k) MDS code to be defined below.
  5) For each transmitted symbol, attach a header to enable decoding at relay r_{j+1}.

To establish the validity of the proposed scheme, we first show the following Lemma.

Lemma 1: If for all $l < j$, for any $j \in \{1, \ldots, L\}$, and any $i \in \mathbb{Z}$, $\mathbf{G}_i^{(r_l)}$ is a generator matrix of an (n_{l+1}, k) MDS code (i.e., that the code used in each previous relay r_l is an (n_{l+1}, k) MDS code) then $\tilde{\mathbf{s}}_i$ can be recovered at time $i+N_1^j+k-1$ in relay r_j assuming r_j knows the structure of code used by r_{j-1}.

Proof: Assuming $N_1, \ldots, N_{L+1}$-erasure sequence means that for any $j \in \{0, \ldots, L\}$

$$e^{j,\infty} \in \Omega_{N_j} \tag{32}$$

i.e., that the maximal number of erasures in link (r_{j-1}, r_j) is N_j. Since we assumed $\mathbf{G}_i^{(r_{j-1})}$ is an (n_j, k) MDS code, and since $n_j = k + N_j$, it follows that it is guaranteed that k symbols out of the n_j transmitted symbols will not be erased. Further, since relay r_{j-1} starts sending the coded symbols at time $i + N_1^{j-1}$ and relay r_j starts forwarding the non-erased symbols received from r_{j-1} at time $i + N_1^{j-1} + N_j$ (after buffering any non-erased symbols from the first N_j coded symbols) it is guaranteed that relay r_j could forward the $k-1$ non-erased coded symbols sent from r_{j-1} by time $i + N_1^j + k - 2$.

In step (3) above, relay r_j needs to recover all k information symbols at time $i + N_1^j + k - 1$. We note that this step is feasible since assuming $\mathbf{G}_i^{(r_{j-1})}$ is the generator matrix of an (n_j, k) MDS code, any of its k information symbols can be

[3] We note that equivalently, the symbol received at time $i + N_1^j + k - 1$ by relay r_j, could also be forwarded and the transmission of re-encoded symbols would start at time $i + N_1^j + k$.

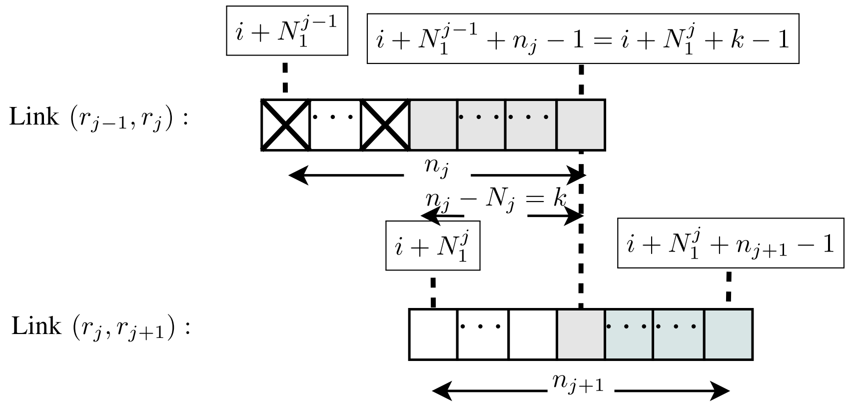

Fig. 5. Two phases of transmission in link (r_j, r_{j+1}). The symbols with white background are symbols forwarded from link (r_{j-1}, r_j). The shaded symbols are transmitted after the k information symbols are decoded. Hence, they are either symbols which were erased in link (r_{j-1}, r_j) or additional (independent) linear combinations of the information symbols.

recovered from any k non-erased symbols. We note that relay r_{j-1} transmits its code at time indices

$$[i + N_1^{j-1}, \dots, i + N_1^{j-1} + n_j - 1]. \tag{33}$$

Therefore, the last symbol of this code is received at relay r_j at time

$$\begin{aligned} i + N_1^{j-1} + n_j - 1 &= i + N_1^{j-1} + T - \sum_{l=1, l\neq j}^{L+1} N_l + 1 - 1 \\ &= i + N_1^{j-1} + N_j + (T - \sum_{l=1}^{L+1} N_l + 1) - 1 \\ &= i + N_1^j + k - 1. \end{aligned} \tag{34}$$

These phases are depicted in Figure 5 below. In this example, the first $n_j - k = N_j$ symbols sent from r_{j-1} are erased in link (r_{j-1}, r_j). Hence, the last k symbols are forwarded as-is by relay r_{j+1} followed by $n_{j+1} - k$ independent linear combination of the information symbols. □

We next discuss the structure associated with the matrices $\mathbf{G}_i^{(r_j)}$ that ultimately establish that it is indeed a generator matrix associated with an (n_{j+1}, k) MDS code as required by the above lemma. First, recall that $n_{\max} = \max_{1 \le l \le L+1} n_l$ (which equals (9)), and let $\mathbf{G}_{\max}$ be the generator matrix associated with $(n_{\max}, k)$ MDS code. Denoting $N_{\max} = \max_j N_j$ and since $n_{\max} = N_{\max} + k$, it follows that the code associated with $\mathbf{G}_{\max}$ can recover from $N_{\max}$ erasures in arbitrary positions.

Proposition 1: Consider any (n_{j+1}, k) block code with a generator matrix that is generated by puncturing $N_{\max} - N_{j+1}$ columns of the generator matrix $\mathbf{G}_{\max}$ and applying a permutation to the remaining columns. Then such a code is also an MDS code.

Proof: The proof follows from the fact that collection of k columns of $\mathbf{G}_{\max}$ are linearly independent and both puncturing and permutation operations will preserve this property. □

Corollary 1: The generator matrix generated at relay r_j i.e., $\mathbf{G}_i^{(r_j)}$ is obtained from $\mathbf{G}_{\max}$ by puncturing $N_{\max} - N_j$ columns and permuting the remaining columns.

Proof: The relay first forwards $k - 1$ symbols as stated in step 2 above. This implies that the first $k-1$ columns of $\mathbf{G}_i^{(r_j)}$ must equal some $k-1$ columns from $\mathbf{G}_{\max}$. After the recovery of the k'th information symbol, relay r_j sends $n_{j+1} - k + 1$ symbols from $\tilde{\mathbf{s}}_i \times \mathbf{G}_{\max}$ that were not sent before by time $i + N_1^j + n_{j+1} - 1$. This is equal to introducing $n_{j+1} - k + 1$ unique columns from $\mathbf{G}_{\max}$ (step (4) above) to $\mathbf{G}_i^{(r_j)}$ which were not used before. □

We can now state the following:

Corollary 2: At each relay r_j, $\mathbf{G}_i^{(r_j)}$ is a generator matrix of an (n_{j+1}, k) MDS code for all j.

Proof: This follows using induction (since from the construction, the generator matrices at the source, $\mathbf{G}_i^{(r_0)}$ are generator matrices of an (n_1, k) MDS code) as well as Lemma 1 and Proposition 1. □

For specific examples on how the generator matrices of the block codes are generated (in different scenarios, i.e., when the code rate increases or decreases), see Appendix C. Next, we show the following Lemma.

Lemma 2 (Based on Lemma 3 in [1]): Suppose $T \ge N$, and let $k \triangleq T - N + 1$ and $n \triangleq k + N$. For any $\mathbb{F}$ such that $|\mathbb{F}| \ge n$, there exists an N-achievable point-to-point $(n, k, T)_{\mathbb{F}}$-block code.

Proof: The proof follows directly from the definitions of the MDS codes. Any (n, k) MDS code is an $(n, k, n-1)$ block code. Thus, any (n, k) MDS code is $(n-k)$-achievable where all symbols can be decoded by the end of the code block. As mentioned in Section I-A, when $|\mathbb{F}| \ge n$ there exists an (n, k) MDS code. Therefore, when $|\mathbb{F}| \ge n$ we conclude that there exists an N-achievable point-to-point $(n, k, T)_{\mathbb{F}}$-block code. □

Recalling that when transmitting $\tilde{\mathbf{s}}_i$, relay r_j starts its transmission at time $i + N_1^j$ we have the following corollary.

Corollary 3: For any $i \in \mathbb{Z}_+$ and for any $N_1, \dots, N_{L+1}$-erasure sequence, the source packet $\tilde{\mathbf{s}}_i$ can be recovered at relay r_{j+1} at time $i + T - \sum_{l=j+2}^{L+1} N_l$.

Proof: From the construction of the code and Corollary 2, we have that relay r_j (for any $j \in \{0, \dots, L\}$) starts transmitting (over the diagonal) the coded symbols of $\tilde{\mathbf{s}}_i$ at time $i + N_1^j$ using an (n_{j+1}, k) MDS code. From Lemma 2 it follows that for any $N_1, \ldots, N_{L+1}$-erasure sequence the code used in each relay r_j to transmit $\tilde{\mathbf{s}}_i$ is N_{j+1}-achievable $(n_{j+1}, k, N_1^j + T_{j+1})_{\mathbb{F}}$ point-to-point block code, i.e., $\tilde{\mathbf{s}}_i$ can be recovered at relay r_{j+1} at time

$$i + N_1^j + T_{j+1} = i + N_1^j + T - \sum_{l=1, l\neq j+1}^{L+1} N_l = i + T - \sum_{l=j+2}^{L+1} N_l. \quad (35)$$

□

We take a closer look at the channel between the last relay and the destination (r_L, r_{L+1}). Following Corollary 3 we have

Corollary 4: For any $i \in \mathbb{Z}_+$ and for any $N_1, \ldots, N_{L+1}$-erasure sequence, $\tilde{\mathbf{s}}_i$ (28) can be decoded at the destination at time

$$i + T - \sum_{l=L+2}^{L+1} N_l = i + T, \quad (36)$$

i.e., using the construction suggested above, $\tilde{\mathbf{s}}_i$ can be decoded at the destination with delay of T for any $N_1, \ldots, N_{L+1}$-erasure sequence.

Next, we show that this Corollary means that the construction suggested above generates an $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$ streaming code which is also $N_1, \ldots, N_{L+1}$-achievable.

Lemma 3: The streaming code resulting from using $\mathbf{G}_i^{(r_j)}$ defined above in each node $j \in \{0, \ldots, L\}$ for every $i \in \mathbb{Z}_+$ is an $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$ streaming code which is also $N_1, \ldots, N_{L+1}$-achievable.

Proof:

From Corollary 4 it follows that for any $i \in \mathbb{Z}_+$, $\tilde{\mathbf{s}}_i$ can be recovered with an overall delay of T. Recalling that $\tilde{\mathbf{s}}_i = [s_i[0] \ s_{i+1}[1] \ \cdots \ s_{i+k-1}[k-1]]$, $s_i[0]$ (which is the first symbol in $\tilde{\mathbf{s}}_i$), can be recovered with an overall delay of T for any $N_1, \ldots, N_{L+1}$-erasure sequence.

Similarly we note that $s_i[1]$ (which is the second symbol in $\tilde{\mathbf{s}}_{i-1}$) can be recovered with an overall delay of $T-1$ and $s_i[k-1]$ (which is the k'th symbol in $\tilde{\mathbf{s}}_{i-k+1}$) can be recovered with an overall delay of $T-k$ for any $N_1, \ldots, N_{L+1}$-erasure sequence. Thus, we conclude that the $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$ streaming code resulting from the construction described above is an $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$ streaming code which is also $N_1, \ldots, N_{L+1}$-achievable. □

It now remains to discuss how the structure of $\mathbf{G}^{(r_{j-1})}$ can be communicated by relay node r_{j-1} to relay node r_j to facilitate the decoding at node r_j. Following Corollary 1, we define the header for each symbol as a number that indicates the location of the column from $\mathbf{G}_{\max}$ that was used to generate this symbol. Thus, the header attached to each transmitted symbol from each relay is a number in the range $[1, \ldots, n_{\max}]$ and can be communicated using $\log n_{\max}$ bits per symbol.

We can now present the proof of Theorem 3.

Proof of Theorem 3: We first note that, as mentioned in Section I-A, assuming $|\mathbb{F}| \geq n_{\max}$ means that an (n_{max}, k) MDS code exists. Following Proposition 1 for any $j \in \{0, \ldots, L\}$ and any $i \in \mathbb{Z}_+$, there exists $\mathbf{G}_i^{(r_j)}$ which is a generator matrix of an (n_{j+1}, k) MDS code (as they can be viewed as a result of puncturing the generator matrix of the (n_{max}, k) MDS code).

Following Lemma 3 it follows that streaming code resulting from using $\mathbf{G}_i^{(r_j)}$ defined above in all nodes $j \in \{0, \ldots, L\}$ for every $i \in \mathbb{Z}_+$ is an $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$ streaming code which is also $N_1, \ldots, N_{L+1}$-achievable. The rate of the code transmitted from r_j, without taking the size of the header into account, is $\frac{k}{n_{j+1}}$. Thus, from Definition 5, the overall rate of transmission is upper bounded by

$$R \leq \min_{j \in \{0,1,\ldots,L\}} \frac{k}{n_{j+1}} = C^+_{T, N_1, \ldots, N_L}. \quad (37)$$

The header attached to each packet sent from relay r_j is generated by stacking the n_{j+1} headers used by each symbol generated from an $(n_{j+1}, k, N_1^j + T_j)_{\mathbb{F}}$ block code which is part of each transmission packet. As we defined above, this header is an integer in $[1, \ldots, n_{\max}]$. Hence, the size of the header attached to each packet is $n_{j+1} \log(n_{\max})$ bits. We further note that the size of the header is upper bounded by $n_{\max} \log(n_{\max})$.

To conclude, node r_j transmits n_{j+1} coded symbols (each taken from field $\mathbb{F}$) along with $n_{j+1} \log(n_{j+1}) \leq n_{\max} \log(n_{\max})$ bits of header to transfer k information symbols (each taken from field $\mathbb{F}$). The overall rate is bounded as

$$\begin{aligned} R &\geq \min_j \frac{k \cdot \log(|\mathbb{F}|)}{n_{j+1} \cdot \log(|\mathbb{F}|) + n_{\max} \lceil \log(n_{\max}) \rceil} \\ &= \frac{T - \sum_{l=1}^{L+1} N_l + 1}{\max_j \left(T - \sum_{l=1, l\neq j}^{L+1} N_l + 1 \right) + \frac{n_{\max} \lceil \log(n_{\max}) \rceil}{\log(|\mathbb{F}|)}} \\ &= \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \left(\sum_{l=1, l\neq j}^{L+1} N_l \right) + 1 + \frac{n_{\max} \lceil \log(n_{\max}) \rceil}{\log(|\mathbb{F}|)}}, \end{aligned} \quad (38)$$

where $n_{\max}$ is defined in (9). □

A. Analyzing the Size of the Header Required by State-Dependent SWDF

The size of the overhead of state-dependent SWDF decreases only logarithmically with the field size, which for practical implementations might result in a noticeable rate loss (which in some cases even suggests that other coding schemes such as straightforward extension of state-independent SWDF may result in a better rate). Examining more carefully the example given in Section I-E where $T = 5, \ N_1 = N_2 = N_3 = 1$ for which we showed that state-independent SWDF achieves $R = 2/3$, we note that the suggested coding scheme achieves $R = \frac{3}{4 + \frac{4\lceil \log(4) \rceil}{\log(|\mathbb{F}|)}}$ which outperforms state-independent SWDF when $|F| > 2^{16}$.

However, as we demonstrate next, as the number of relays increases, the required field size for state-dependent SWDF to outperform the straightforward extension of state-independent SWDF gets smaller. For simplicity, we analyze the case of having an odd number of relays $(L = 2u - 1, \ u \in \mathbb{Z}^+)$. We first note that when using straightforward extension of state-independent SWDF, the delay of different symbols is "reset" every two links (i.e., transmission at relays r_2, r_4 and so on can be treated as a transmission from the source). Thus, the optimal rate of straightforward extension of state-independent SWDF can be found by optimizing the delay allocated to each such three-node network.

We denote with $R_m^{\text{SI}}(T_m, N_{2m-1}, N_{2m})$ the rate achieved by state-independent SWDF over the m'th three node network. We thus have

$$R_m^{\text{SI}}(T_m, N_{2m-1}, N_{2m}) = \frac{T_m - N_{2m-1} - N_{2m} + 1}{T_m - \min(N_{2m-1}, N_{2m}) + 1} \triangleq \frac{k_m^{\text{SI}}}{n_m^{\text{SI}}},$$

for $1 \leq m \leq \frac{L+1}{2}$. To formally derive the rate of state-independent SWDF we define concatenation of two streaming codes.

Definition 9: A concatenation of an $(n'_1, \ldots, n'_{L+1}, k', T')_{\mathbb{F}}$ streaming code with an $(n''_1, \ldots, n''_{L+1}, k'', T'')_{\mathbb{F}}$ streaming code is an $(n'_1 + n''_1, \ldots, n'_{L+1} + n''_{L+1}, k' + k'', [T', T''])_{\mathbb{F}}$ streaming code with the following properties

- Let $\mathbf{s} = [\mathbf{s'}_i \ \mathbf{s''}_i]^T$ be the input to the concatenated code where $s'_i \in \mathbb{F}^{k'}$ and $s''_i \in \mathbb{F}^{k''}$.
- Let $\{f_t^{(r_0)'}\}$ be the encoding functions for node r_0 of the first code and $\{f_t^{(r_0)''}\}$ be the encoding functions for node r_0 of the second code. The encoding function of the concatenated code outputs

$$\left[f_t^{(1)'}(s'_0, \ldots, s'_t) \ f_t^{(1)''}(s''_0, \ldots, s''_t)\right]^T.$$

- Let $\mathbf{y}_t^{(r_j)}$ denote the input to relay r_j. Let $\{f_t^{(r_j)'}\}$ be the relaying functions for node r_j of the first code and $\{f_t^{(r_j)''}\}$ be the relaying functions for node r_j of the second code. The relaying function of the concatenated code outputs

$$\left[f_t^{(r_j)'}(\mathbf{y}_0^{(r_j)}, \ldots, \mathbf{y}_t^{(r_j)}) \ f_t^{(r_j)''}(\mathbf{y}_0^{(r_j)}, \ldots, \mathbf{y}_\mathbf{t}^{(\mathbf{r_j})})\right]^T.$$

- Let $\mathbf{y}_t^{(r_{L+1})}$ denote the input to the decoder of the concatenated code. Denote $\{\varphi'_t\}$ as the decoding functions of the first code and $\{\varphi''_t\}$ as the decoding functions of the second code. The output of the concatenated code is

$$\hat{\mathbf{s}}_t = \Bigg[\underbrace{\varphi_{i+T'}(\mathbf{y}_0^{(r_{L+1})}, \ldots, \mathbf{y}_{t+T'}^{(r_{L+1})})}_{\hat{\mathbf{s}'}_t} \ \underbrace{\varphi_{i+T''}(\mathbf{y}_0^{(r_{L+1})}, \ldots, \mathbf{y}_{t+T''}^{(r_{L+1})})}_{\hat{\mathbf{s}''}_t}\Bigg]^T.$$

Following Definition 5, we note that the rate of the concatenated code is $\frac{k'+k''}{\max_j(n'_j+n''_j)}$. Concatenation of M identical $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$ streaming codes results in a streaming code with rate $\frac{Mk}{\max_j(Mn_j)}$.

In a network with $L = 2u - 1$, $u \in \mathbb{Z}^+$ relays, and a delay allocation of $\{T_1, \ldots, T_m\}$, for each $m \in \{1, \ldots, \frac{L+1}{2}\}$, we analyze the rate resulting from a concatenation of $\prod_{i=1, i\neq m}^{\frac{L+1}{2}} k_i^{\text{SI}} = k_1^{\text{SI}} \cdot k_2^{\text{SI}} \cdots k_{m-1}^{\text{SI}} \cdot k_{m+1}^{\text{SI}} \cdots k_{\frac{L+1}{2}}^{\text{SI}}$ identical state-independent SWDF codes used over each three-node network composed of nodes $\{r_{2m-2}, r_{2m-1}, r_{2m}\}$.

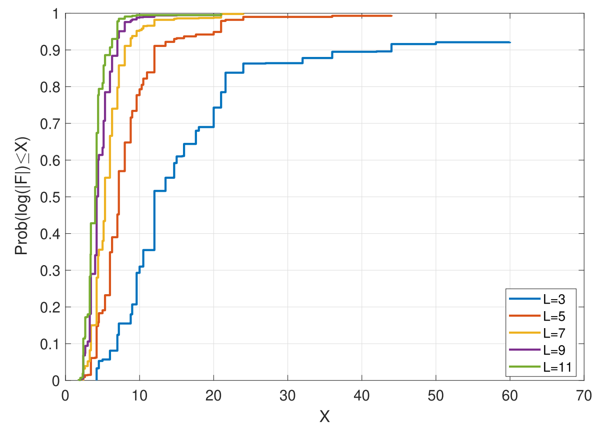

Fig. 6. CDF of $\log(|F|)$ required for state-dependent SWDF to outperform state-independent SWDF for different number of relays.

Using the suggested concatenation scheme means that each state-independent SWDF code, used over the m'th three-node network, transmits $\prod_{m=1}^{\frac{L+1}{2}} k_m^{\text{SI}}$ information symbols in each channel use. Hence, from definition 5 we have

$$R^{\text{SI-SWDF}}(T_1, \ldots, T_m) = \frac{\prod_{m=1}^{L+1} k_m^{\text{SI}}}{\max\left(n_1 \prod_{m\neq 1} k_m^{\text{SI}}, \ldots, n_{\frac{L+1}{2}} \prod_{m\neq \frac{L+1}{2}} k_m^{\text{SI}}\right)} = \min\left(\frac{k_1^{\text{SI}}}{n_1^{\text{SI}}}, \ldots, \frac{k_{\frac{L+1}{2}}^{\text{SI}}}{n_{\frac{L+1}{2}}^{\text{SI}}}\right) = \min_m \left(R_m^{\text{SI}}(T_m, N_{2m-1}, N_{2m})\right).$$

Thus, we have

$$R^{\text{SI-SWDF}} = \max_{\sum_{m=1}^{\frac{L+1}{2}} T_m \leq T} R^{\text{SI-SWDF}}(T_1, \ldots, T_m) = \max_{\sum_{m=1}^{\frac{L+1}{2}} T_m \leq T} \min_m \left(R_m^{\text{SI}}(T_m, N_{2m-1}, N_{2m})\right). \tag{39}$$

For example, in case of a five-node network (a network with three relays), with maximum of N_1, N_2, N_3, N_4 erasures on each link and a total delay of $T \geq \sum_{j=1}^{4} N_j$ we have

$$R_1^{\text{SI}}(T_1, N_1, N_2) = \frac{T_1 - N_1 - N_2 + 1}{T_1 - \min(N_1, N_2) + 1}$$

$$R_2^{\text{SI}}(T_2, N_3, N_4) = \frac{T_2 - N_3 - N_4 + 1}{T_2 - \min(N_3, N_4) + 1}$$

$$R^{\text{SI-SWDF}} = \max_{T_1+T_2 \leq T} \min\left(R_1^{\text{SI}}(T_1, N_1, N_2), R_2^{\text{SI}}(T_2, N_3, N_4)\right).$$

Figure 6 depicts the cumulative distribution function (CDF) of $\log(|F|)$ required for state-dependent SWDF to outperform state-independent SWDF as a function of the number of relays. We randomly draw the maximal number of erasures on each link (uniformly between 1 and 5). The overall delay

TABLE VII
AN EXAMPLE OF ONE MDS CODE TRANSMITTED IN RELAY r_j WHEN CONCATENATION OF M STREAMING CODES IS USED

Time: $i+N_1^j$	$i+N_1^j+1$	$\cdots$	$i+N_1^j+n_{j+1}-1$
$[\tilde{\mathbf{s}}_i^1 \times \mathbf{G}_i^{(r_j)}][0]$			
	$[\tilde{\mathbf{s}}_i^1 \times \mathbf{G}_i^{(r_j)}][1]$		
		$\ddots$	
			$[\tilde{\mathbf{s}}_i^1 \times \mathbf{G}_i^{(r_j)}][n_{j+1}-1]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$[\tilde{\mathbf{s}}_i^m \times \mathbf{G}_i^{(r_j)}][0]$			
	$[\tilde{\mathbf{s}}_i^m \times \mathbf{G}_i^{(r_j)}][1]$		
		$\ddots$	
			$[\tilde{\mathbf{s}}_i^m \times \mathbf{G}_i^{(r_j)}][n_{j+1}-1]$

constraint is the sum of the maximal number of erasures over all links plus another random variable drawn uniformly between 1 and 10. As can be seen, as the number of relays increases (Up to having 11 relays), the required field size for the suggested coding scheme to outperform state-independent SWDF decreases.

Another possible avenue for practical implementation of state-dependent SWDF (while keeping the field size used by the code small) is to concatenate several instances of the same state-dependent code (each using a small field size). We further suggest using the same block code (per diagonal) for all streaming codes at each relay which means that the concatenated streaming code transmits $M \cdot k$ information symbols at each time instance and that the channel packet size transmitted from relay r_j is $M \cdot n_{j+1}$ (where k and n_{j+1} are defined in (26)). Specifically, denoting with

$$\tilde{\mathbf{s}}_i^m = \Big[s_i[(m-1)\cdot k+0] \; s_{i+1}[1] \; \cdots \; s_{i+k-1}[(m-1)\cdot k+k-1]\Big], \quad (40)$$

the information symbols generated at time i and used by the m'th code, an example of the diagonal interleaving of the concatenated codes is given in Table VII below. Therefore, each transmitted packet at relay r_j is composed of $M \cdot n_{j+1}$ symbols.

We thus have the following Lemma.

Lemma 4: When a concatenation of M identical codes is used, the achievable rate of state-dependent SWDF is

$$R \geq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \left(\sum_{l=1, l\neq j}^{L+1} N_l\right) + 1 + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{M\cdot \log(|\mathbb{F}|)}}. \quad (41)$$

Proof: Since all the concatenated codes will experience exactly the same erasures (per link) during transmission, a single header is enough to generate $\mathbf{G}_i^{(r_j)}$ at each relay r_j (where the same $\mathbf{G}_i^{(r_j)}$ is used for all M concatenated codes). Thus, Corollary 2 and Lemma 3 holds for each of the concatenated codes. Recalling (26) and (9) we therefore have

$$\begin{aligned} R &\geq \min_j \frac{M\cdot k \cdot \log(|\mathbb{F}|)}{M\cdot n_{j+1}\cdot \log(|\mathbb{F}|) + n_{\max}\lceil \log(n_{\max})\rceil} \\ &= \frac{M(T - \sum_{l=1}^{L+1} N_l + 1)}{M(T - \min_j \left(\sum_{l=1, l\neq j}^{L+1} N_l\right) + 1) + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}} \\ &= \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \left(\sum_{l=1, l\neq j}^{L+1} N_l\right) + 1 + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{M\cdot \log(|\mathbb{F}|)}}. \end{aligned}$$

□

Note that the concatenation of M independent copies of an $(n,k)_{\mathbb{F}}$ state-dependent SWDF code over field $\mathbb{F}$ or a single (n,k) code over a field $\mathbb{F}^M$, leads to the same length of channel packet — $Mn\log|\mathbb{F}|$ bits. Both approaches will lead to the same overhead, although the former may be desirable in practice as it involves code operations over a smaller field.

IV. AN UPPER BOUND ON THE LOSS PROBABILITY ATTAINED BY STATE-DEPENDENT SWDF FOR I.I.D. RANDOM ERASURES

In Section III, the state-dependent SWDF scheme was described, and a lower bound on its achievable rate was derived while assuming a deterministic erasure model. In this section, we develop an upper bound on the average loss probability when this scheme is applied over channels with random (i.i.d.) erasures.

Let $s_i[0], s_i[1], \ldots, s_i[k-1]$ be the k source symbols transmitted by node r_0 at each discrete time i. We note that for the (n_1, k) MDS codes used by the sender, the following property holds:

- For every $s_i[v]$ located at the $(v+1)$th position of the length-k packet transmitted at time i by the $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$ streaming code over (r_0, r_1), $\hat{s}_i^{(r_1)}[v]$ is generated by the relay, at the latest, at time $i - v + n_1 - 1$ (i.e., after transmission of n_1 symbols from r_0). If there are at most N_1 erasures inside the window $\{i-v, i-v+1, \ldots, i-v+n_1-1\}$, then $\hat{s}_i^{(r_1)}[v] = s_i[v]$.

Hence, $\mathbf{s}_i$ can be fully recovered at relay r_1 if for all $v \in \{0, 1, \ldots, k-1\}$, in any window $\{i-v, i-v+1, \ldots, i-v+n_1-1\}$, there are at most N_1 erasures. We bound the loss probability by analyzing the probability in which in the window $\{i-k+1, i-k+2, \ldots, i+n_1-1\}$ there are at most N_1 erasures.

Since the state-dependent SWDF encode the same information symbols (per diagonal) in each relay, we note that in the general case, when transmitting the $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$ streaming code over (r_{j-1}, r_j):

- $\hat{\mathbf{s}}_i^{(r_j)}[v]$ is generated by the relay, at the latest, at time $i+N_1^{j-1}-v+n_j-1$ (i.e., after transmission of n_j symbols from relay r_{j-1}). If there are at most N_j erasures inside the window $\{i+N_1^{j-1}-v, i-v+1, \ldots, i+N_1^{j-1}-v+n_j-1\}$, $\hat{s}_i^{(r_j)}[v] = s_i[v]$.

Hence, $\mathbf{s}_i$ can be fully recovered at relay r_j if for all $v \in \{0, 1, \ldots, k-1\}$, in any window $\{i+N_1^{j-1}-v, i+N_1^{j-1}-v+1, \ldots, i+N_1^{j-1}-v+n_j-1\}$ there are at most N_j erasures. Similar to [1], we bound the loss probability by analyzing the probability in which in the window $\{i+N_1^{j-1}-k+1, i+N_1^{j-1}-k+1, \ldots, i+N_1^{j-1}+n_j\}$ there are at most N_j erasures.

Denoting the average Loss probability as

$$P_{T,N_1,N_2,\ldots,N_{L+1}} \triangleq \lim_{M\to\infty} \frac{1}{M} \Pr\{\hat{\mathbf{s}}_i \neq \mathbf{s}_i\} \tag{42}$$

achieved by the above state-dependent SWDF strategy under the random erasure model. Define $\alpha_j = \Pr(e_0^j = 1)$ to be the erasure probability in link (r_{j-1}, r_j). According to the achievability conditions we have

$$\Pr\left(\hat{\mathbf{s}}_i \neq \mathbf{s}_i \;\middle|\; \sum_{u=i-k+1}^{i+n_1+1} e_u^1 \le N_1, \sum_{u=i-k+1}^{i+n_2+1} e_u^2 \le N_2, \ldots, \sum_{u=i-k+1}^{i+n_{j+1}+1} e_u^{L+1} \le N_{L+1}\right) = 0 \tag{43}$$

for every $i \ge T - N_1^{L+1}$. Since

$$\begin{aligned}
&\Pr\Bigg(\left\{\sum_{u=i-k+1}^{i+n_1+1} e_u^1 > N_1\right\} \bigcup \left\{\sum_{u=i-k+1}^{i+n_2+1} e_u^2 > N_2\right\} \bigcup \cdots \\
&\bigcup \left\{\sum_{u=i-k+1}^{i+n_{L+1}+1} e_u^{L+1} > N_{L+1}\right\}\Bigg) \\
&\le \sum_{u=N_1+1}^{2k+2N_1+1} \binom{2k+2N_1+1}{u} (\alpha_1)^u (1-\alpha_1)^{2k+2N_1+1-u} \\
&\quad + \sum_{u=N_2+1}^{2k+2N_2+1} \binom{2k+2N_2+1}{u} (\alpha_2)^u (1-\alpha_2)^{2k+2N_2+1-u} + \ldots \\
&\quad + \sum_{u=N_{L+1}+1}^{2k+N_{L+1}+1} \binom{2k+2N_{L+1}+1}{u} (\alpha_{L+1})^u (1-\alpha_{L+1})^{2k+2N_{L+1}+1-u}.
\end{aligned} \tag{44}$$

Denoting

$$\kappa_j(T, N_1, \ldots, N_{L+1}) = (2k+N_j+1) \max_{u\in\{N_j+1,\cdots,2k+2N_j+1\}} \binom{2k+2N_j+1}{u},$$

it follows that

$$\begin{aligned}
P_{T,N_1,N_2,\ldots,N_{L+1}} \le\; &\kappa_1(T, N_1, \ldots, N_{L+1}) \cdot (\alpha_1)^{N_1+1} + \\
&\kappa_2(T, N_1, \ldots, N_{L+1}) \cdot (\alpha_2)^{N_2+1} + \ldots \\
&+ \kappa_{L+1}(T, N_1, \ldots, N_{L+1}) \cdot (\alpha_{L+1})^{N_{L+1}+1}.
\end{aligned} \tag{45}$$

where κ_j does not depend on α_j (or on any other α_k for any $k \neq j$). Hence, $P_{T,N_1,N_2,\ldots,N_{L+1}}$ decays exponentially fast in $\min(N_1+1, N_2+1, \ldots, N_{L+1}+1)$.

Remark 6: We note that in the derivation of the upper bound, we required a maximum of N_j erasures in windows of size $n_j + k$ channel packets. When $n_j + k > T$ this is a loose upper bound since we know that the code can recover from any N_j erasures in a window of size $T+1$ channel packets.

V. Numerical Results

In this section, we show the performance of the state-dependent SWDF scheme on random (i.i.d.) erasure channels (described in Section IV). We consider $L+1$-node relay network where i.i.d. erasures are independently introduced to all channels. We denote with α_j the probability of experiencing an erasure in each time slot for channel (r_{j-1}, r_j).

Similar to [1], we will compare state-dependent SWDF with message-wise decode and forward (DF) and instantaneous forwarding, which we briefly recall. In message-wise DF, all the symbols in the same source message are decoded by relay r_j subject to the delay constraint T_j such that $\sum_j T_j \le T$. The overall rate of message-wise DF is

$$R^{\text{Message}}_{T,N_1,N_2,\ldots,N_{L+1}} = \max_{(T_1,\ldots,T_{L+1}):\sum_j T_j \le T} \min\left(C_{T_1,N_1}, C_{T_2,N_2}, \ldots, C_{T_{L+1},N_{L+1}}\right). \tag{46}$$

More precisely, we consider message-wise DF scheme constructed by concatenating $L+1$ streaming codes where the j'th code is an $(n_j^{\text{Message}}, k, T_j)_{\mathbb{F}}$-streaming code.

We also consider an instantaneous forwarding (IF) strategy, which uses a point-to-point streaming code over the $L+1$-node relay network as if the network is a point-to-point channel. More specifically, under the IF strategy, the source transmits symbols generated by the streaming code and relay r_j forwards every symbol received from relay r_{j-1} in each time slot. The overall point-to-point channel induced by the IF strategy experiences an erasure if either one of the channels experiences an erasure. This results in rate

$$R^{\text{IF}} = C_{T,\sum_l N_l}. \tag{47}$$

Ideally, in order to properly compare the different schemes, it is desirable to set exactly the same parameters (i.e., same total delay constraint and the same transmission rate) to all the coding schemes. Unfortunately, this cannot be done as fixing one parameter results in different parameter setting for different schemes. As a result, we fix one parameter at a time and then find the best code parameters that are feasible with respect to the other, as explained below. We note in advance that we are considering a packet erasure channel that either erases the entire transmitted packet or successfully transmits it.

Thus, we do not include the difference in field-size of different codes in our simulations.

In our simulations, we study the error-correcting capabilities of all schemes in case of having two relays with two different constraints (which converge for the case of state-dependent SWDF)

- $R = 2/3$ and $T \leq 9$.
- $R \geq 2/3$ and $T = 9$.

We further simulate a symmetric topology, i.e., we assume the same error probability for all segments. Since we assume symmetric topology, we focus on schemes that have the same error-correcting capabilities for all segments.

- State-dependent SWDF can support $(N_1, N_2, N_3) = 2$ erasures in each segment with total delay of $T = 9$ which is constructed using three $(6,4,5)_{\mathbb{F}}$ streaming code. While the rate of the code is strictly lower than $2/3$ due to the overhead it uses, as we noted above, it approaches $2/3$ as $|\mathbb{F}|$ increases.
- Message-wise DF
  - $R = 2/3,\ T \leq 9$: MWDF can support rate of $2/3$ with $(N_1, N_2, N_3) = 1$ erasures in each segment and $T_1 = T_2 = T_3 = 2$ (which results in $T = 6$) which is constructed using $(3,2,2)_{\mathbb{F}}$ streaming codes. As mentioned in [12], higher rate codes (such as $(4,3,3)_{\mathbb{F}}$) are excluded since $(3,2,2)_{\mathbb{F}}$ can correct more erasure patterns.
  - $T = 9,\ R \geq 2/3$: With $T = 9$, we show the performance of Message-wise DF with $(N_1, N_2, N_3) = 1$ and $T_1 = T_2 = T_3 = 3$ which results in $R = 3/4$ which is constructed using $(4,3,3)_{\mathbb{F}}$ streaming codes.
- IF
  - $R = 2/3,\ T \leq 9$: to achieve $R = 2/3$, we set $T = 8$ and $\sum_j N_j = 3$. Hence, we simulated IF using a $(9,6,8)_{\mathbb{F}}$ streaming code.
  - $T = 9,\ R \geq 2/3$: When $T = 9$ we note that since $C_{9,\sum_j N_j} = \frac{10-\sum_j N_j}{10}$, requiring $C_{9,\sum_j N_j} \geq 2/3$ means that $\sum_j N_j \leq 3$. Hence, we simulated IF using a $(10,7,9)_{\mathbb{F}}$ streaming code.

In Figure 7 we plot the frame loss ratio for state-dependent SWDF, Message-wise DF and for IF. We further plot the upper bound (using Equation (44)) for State-dependent SWDF derived in Section IV with $R = 2/3$, $T \leq 9$ and $N_1 = N_2 = N_3 = 2$ while assuming $\forall j : \alpha_j = \alpha$.

Since enforcing rate $2/3$ results in a low overall delay ($T = 6$ for example, in case of message-wise DF), we further plot the performance of all schemes where we force $T = 9$ and allow the rate to be greater than or equal to $2/3$ (while trying to find the lowest rate possible). Figure 8 depicts all schemes when the rates are as close as possible to $2/3$ (from above) with $T = 9$.

Finally, we note that our simulations cannot provide a direct comparison between the performance of difference codes, as they cannot be set to have identical parameters. Nevertheless, we believe that they provide some useful intuition on the behavior of different schemes considered in the paper.

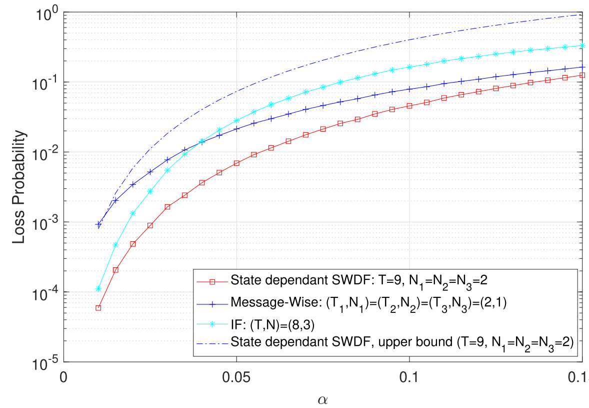

Fig. 7. Four-node (two relays) network loss probability for state-dependent SWDF, message-wise DF and IF with $T \leq 9$, rate $2/3$ and largest $N_1 + N_2 + N_3$ where α denotes the erasure probability (same over all hops).

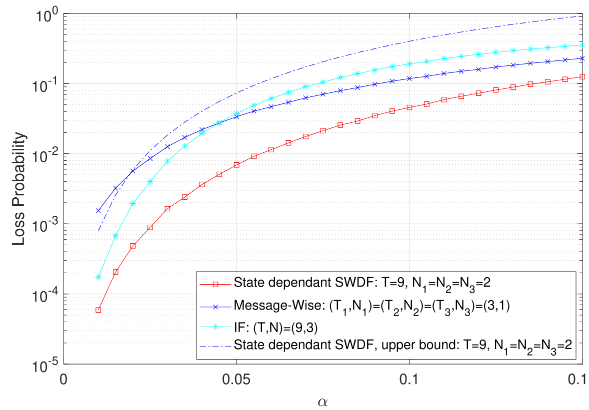

Fig. 8. Four-node (two relays) network loss probability for state-dependent SWDF, message-wise DF and IF with delay $T = 9$, $R \geq 2/3$ and largest $N_1 + N_2 + N_3$ where α denotes the erasure probability (same over all hops).

VI. Extension to Sliding Window Model

Consider the following sliding window model. For each $j \in \{1, \ldots, L+1\}$, channel (r_{j-1}, r_j) introduces at most N_j erasures in any period of $T+1$ consecutive time slots (sliding window of size $T+1$).

Assuming the sliding window model described above, denote with $C^{\text{sw}}_{T-\sum_{l=2}^{L+1} N_l, N_1}$ as the upper bound on the achievable rate for $(N_1, N_2, \ldots, N_{L+1})$ channel. We further denote R^{sw} as the achievable rate assuming the sliding window model.

Our goal is to show that

$$C^{\text{sw}}_{T-\sum_{l=2}^{L+1} N_l, N_1} \leq C_{T-\sum_{l=2}^{L+1} N_l, N_1} \tag{48}$$

and

$$R^{\text{sw}} = \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \left(\sum_{l=1, l\neq j}^{L+1} N_l\right) + 1 + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}}. \tag{49}$$

With respect to the upper bound, since for any $j \in \{1, \ldots, L+1\}$, N_j-erasure sequence can be introduced by channel (r_{j-1}, r_j) in the sliding window model (48) holds.

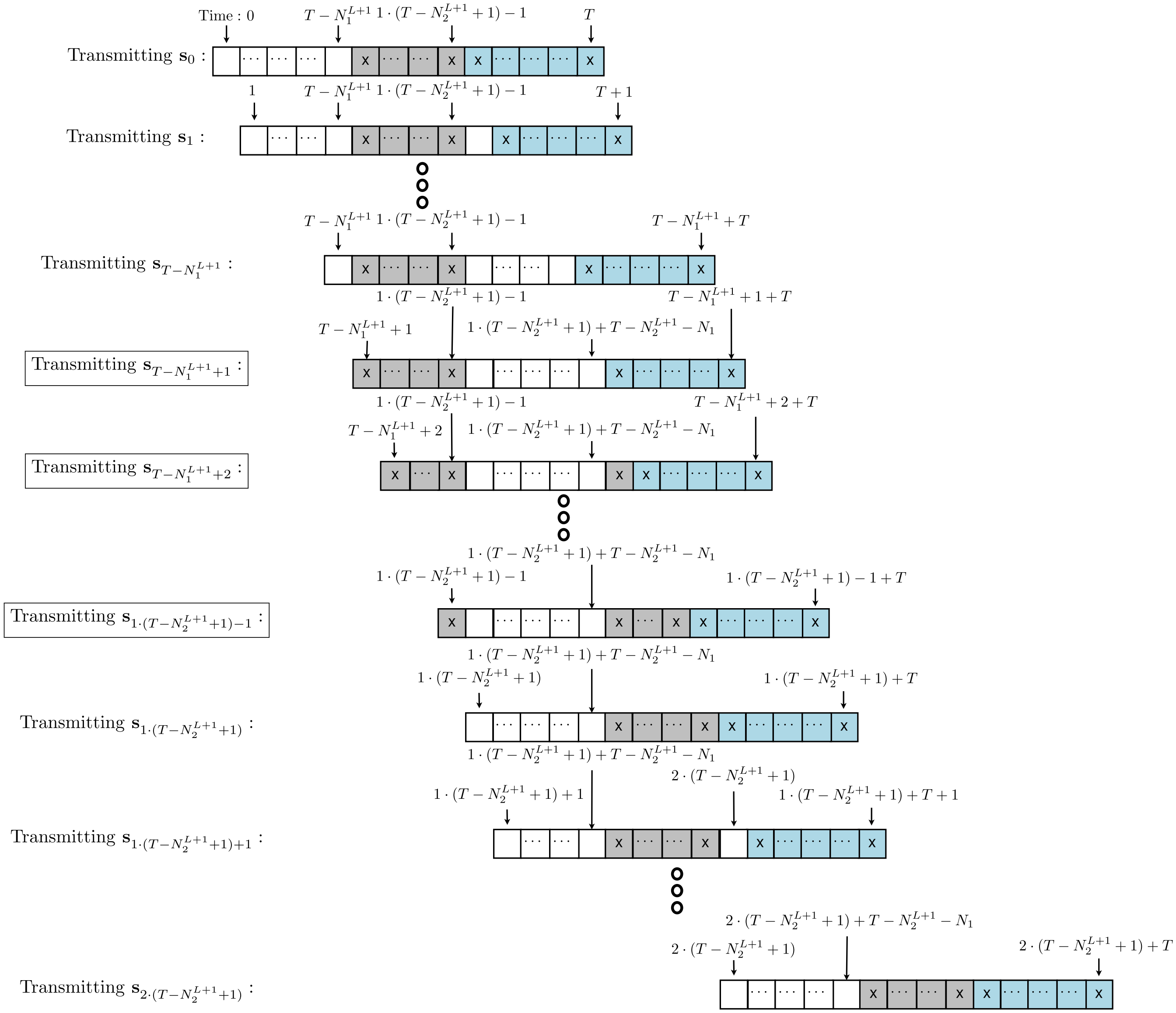

Fig. 9. Suggested erasure pattern for the first link. Light black time indices are erasures which occur on the other links (and thus preclude using these time indices in recovering the information message) which is described in (18). Light blue time indices are erasures which occur on the first link. Each information symbol transmitted at time i is recovered from the channel packets in time indices marked in white, which is described in (50). The maximal delay for recovery of a packet which was sent at time i and was erased in the first link (transmission marked with solid box) is $T - N_2^{L_1}$.

Next, we show that the state-dependent scheme can achieve the same rate under the sliding window model. As was shown in Section III, for any $j \in \{1, \ldots, L+1\}$, each symbol can be recovered as long as channel (r_{j-1}, r_j) introduces at most N_j erasures in a window of size n_j. From (26) we have $n_j < T + 1$ for all $j \in \{1, \ldots, L+1\}$. Hence, it follows that all $L + 1$ conditions hold under the sliding window model, thus the state-dependent SWDF code can recover all symbols and hence (49) holds.

VII. Concluding Remarks

This paper proposes a new coding scheme, state-dependent symbol-wise decode and forward, for streaming codes over a multi-hop relay network consisting of L relay nodes. We analyze the rate achieved over a class of packet-erasure networks where each communication link is subjected to a certain maximum number of erasures. In the special case when the communication link between the source and the first relay has the maximum number of erasures, we establish the optimality of the proposed link. Our proposed scheme requires that the packets transmitted at each relay node r_j be a function of the erasure sequence on the link between relay nodes r_{j-1} and r_j. This is in contrast to prior work [12] on the three node relay setting, where a state-independent coding scheme was proposed. Our proposed scheme can recover the rate proposed in [12] for the three-node setup as the field-size increases. Furthermore, to the best of our knowledge, the scheme in [12] could not be easily extended to more than three-nodes. In contrast, our “state-dependent” coding scheme can be naturally extended to any arbitrary number of relay nodes. Although our analysis focuses on a simplistic erasure model, in section IV showed that the error probability of the proposed coding scheme could be upper bounded for an i.i.d. erasure channel, and Section V provided simulation results showing that the proposed coding scheme exhibits promising performance over other simple options such as message-wise DF and instantaneous forwarding.

APPENDIX A
SPECIFIC ERASURE PATTERN USED IN DERIVATION OF THEOREM 2

The overall erasure pattern is depicted in Figure 9 where time indices shaded with light black are the erasures on the other links (thus these channel packets cannot be used to recover the information message) and time indices shaded with gray are erasures on the first link. We therefore have, (50), shown at the bottom of the page, where the conditional entropy involves $j(T-N_2^{L+1}+1)+1$ source messages and $(j+1)(T-N_1^{L+1}+1)$ channel packets.

APPENDIX B
DERIVATIONS IN PROOF OF THEOREM 2

Derivation of (21)*:* Since the $(N_1,\dots,N_{L+1})$-achievable $(n_1,\dots,n_{L+1},k,T)_{\mathbb{F}}$-streaming code restricted to channel (r_0,r_1) can be viewed as a point-to-point streaming code with rate k/n_1 and delay $T-N_2^{L+1}$ which can correct the periodic erasure sequence $\tilde{e}^{\infty}$ illustrated in Figure 4, it follows from the arguments in [11] Section IV-A that (21) holds. For the sake of completeness, we present a rigorous proof below.

Using (19), we have

$$|\mathbb{F}|^{k\times[j(T-N_2^{L+1}+1)+1]} \le |\mathbb{F}|^{n\times(j+1)(T-N_1^{L+1}+1)} \tag{52}$$

because $j(T-N_2^{L+1}+1)+1$ source messages can take $|\mathbb{F}|^{k\times[j(T-N_2^{L+1}+1)+1]}$ values and $(j+1)(T-N_1^{L+1}+1)$ channel packets can take at most $|\mathbb{F}|^{n\times(j+1)(T-N_1^{L+1}+1)}$ values for each j. Taking logarithm on both sides of (52) followed by

$$\begin{aligned}
&H\left(\mathbf{s}_0 \,\middle|\, \left\{\mathbf{x}_0^{(r_0)},\mathbf{x}_1^{(r_0)},\dots,\mathbf{x}_{T-N_2^{L+1}-N_1}^{(r_0)}\right\}\right)=0\\
&H\left(\mathbf{s}_1 \,\middle|\, \left\{\mathbf{x}_1^{(r_0)},\mathbf{x}_2^{(r_0)},\dots,\mathbf{x}_{T-N_2^{L+1}-N_1}^{(r_0)},\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)}^{(r_0)}\right\},\mathbf{s}_0\right)=0\\
&\qquad\vdots\\
&H\left(\mathbf{s}_{T-N_2^{L+1}-N_1} \,\middle|\, \left\{\mathbf{x}_{T-N_2^{L+1}-N_1}^{(r_0)},\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)}^{(r_0)},\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)+1}^{(r_0)},\dots,\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)+T-N_2^{L+1}-N_1-1}^{(r_0)}\right\},\mathbf{s}_0,\dots,\mathbf{s}_{T-N_2^{L+1}-N_1-1}\right)\\
&=0\\
&H\left(\mathbf{s}_{T-N_2^{L+1}-N_1+1} \,\middle|\, \left\{\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)}^{(r_0)},\dots,\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)+T-N_2^{L+1}-N_1}^{(r_0)}\right\},\mathbf{s}_0,\dots,\mathbf{s}_{T-N_2^{L+1}-N_1}\right)=0\\
&H\left(\mathbf{s}_{T-N_2^{L+1}-N_1+2} \,\middle|\, \left\{\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)}^{(r_0)},\dots,\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)+T-N_2^{L+1}-N_1}^{(r_0)}\right\},\mathbf{s}_0,\dots,\mathbf{s}_{T-N_2^{L+1}-N_1+1}\right)=0\\
&\qquad\vdots\\
&H\left(\mathbf{s}_{T-N_2^{L+1}} \,\middle|\, \left\{\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)}^{(r_0)},\dots,\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)+T-N_2^{L+1}-N_1}^{(r_0)}\right\},\mathbf{s}_0,\dots,\mathbf{s}_{T-N_2^{L+1}-1}\right)=0\\
&H\left(\mathbf{s}_{T-N_2^{L+1}+1} \,\middle|\, \left\{\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)}^{(r_0)},\dots,\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)+T-N_2^{L+1}-N_1}^{(r_0)}\right\},\mathbf{s}_0,\dots,\mathbf{s}_{T-N_2^{L+1}}\right)=0\\
&H\left(\mathbf{s}_{T-N_2^{L+1}+2} \,\middle|\, \left\{\mathbf{x}_{1\cdot(T-N_2^{L+1}+2)}^{(r_0)},\dots,\mathbf{x}_{1\cdot(T-N_2^{L+1}+1)+T-N_2^{L+1}-N_1}^{(r_0)},\mathbf{x}_{2\cdot(T-N_2^{L+1}+1)}^{(r_0)}\right\},\mathbf{s}_0,\dots,\mathbf{s}_{T-N_2^{L+1}+1}\right)=0\\
&\qquad\vdots\\
&H\left(\mathbf{s}_{2\cdot(T-N_2^{L+1}+1)} \,\middle|\, \left\{\mathbf{x}_{2\cdot(T-N_2^{L+1}+1)}^{(r_0)},\dots,\mathbf{x}_{2\cdot(T-N_2^{L+1}+1)+T-N_2^{L+1}-N_1}^{(r_0)}\right\},\mathbf{s}_0,\dots,\mathbf{s}_{2\cdot(T-N_2^{L+1}+1)-1}\right)=0\\
&\qquad\vdots
\end{aligned} \tag{50}$$

Using the chain rule, we have

$$\begin{aligned}
&H\left(\mathbf{s}_0,\dots,\mathbf{s}_{T-N_2^{L+1}+1} \,\middle|\, \left\{\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)}^{(r_0)},\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)+1}^{(r_0)},\dots,\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)+T-N_1-N_2^{L+1}}^{(r_0)}\right\}_{m=0}^{1}\right)=0\\
&H\left(\mathbf{s}_0,\dots,\mathbf{s}_{2\cdot(T-N_2^{L+1}+1)} \,\middle|\, \left\{\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)}^{(r_0)},\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)+1}^{(r_0)},\dots,\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)+T-N_1-N_2^{L+1}}^{(r_0)}\right\}_{m=0}^{2}\right)=0\\
&\qquad\vdots
\end{aligned}$$

Thus, for each $j\in\mathbb{N}$ we have

$$H\left(\mathbf{s}_0,\dots,\mathbf{s}_{j\cdot(T-N_2^{L+1}+1)} \,\middle|\, \left\{\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)}^{(r_0)},\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)+1}^{(r_0)},\dots,\mathbf{x}_{m\cdot(T-N_2^{L+1}+1)+T-N_1-N_2^{L+1}}^{(r_0)}\right\}_{m=0}^{j}\right)=0, \tag{51}$$

TABLE VIII

EXAMPLE OF INCREASING THE RATE BETWEEN LINKS. IN THIS EXAMPLE, $N_{j+1} = 2,\ N_{j+2} = 1,\ T' = 4$, HENCE $\frac{k}{n_{j+2}} = 2/4 < 2/3 = \frac{k}{n_{j+1}}$. ASSUMING SYMBOL $i + N_1^j$ AND $i + N_1^j + 2$ WERE ERASED WHEN TRANSMITTED IN LINK (r_j, r_{j+1}). PARITY SYMBOL ARE SHADED

$i+N_1^j$	$i+N_1^j+1$	$i+N_1^j+2$	$i+N_1^j+3$	$i+N_1^j+4$
Link (r_j, r_{j+1})				
a_i				
	b_{i+1}			
		$f^1(a_i, b_{i+1})$		
			$f^2(a_i, b_{i+1})$	
Link (r_{j+1}, r_{j+2})				
		b_{i+1}		
			a_i	
				$f^1(a_i, b_{i+1})$

TABLE IX

EXAMPLE OF REDUCING RATE BETWEEN NODES. IN THIS EXAMPLE, $N_{j+1} = 1,\ N_{j+2} = 2,\ T' = 4$, HENCE $\frac{k}{n_{j+1}} = 2/3 > 2/4 = \frac{k}{n_{j+2}}$. ASSUMING SYMBOL $i + N_1^j$ WAS ERASED WHEN TRANSMITTED IN LINK (r_j, r_{j+1}). PARITY SYMBOLS ARE SHADED

$i+N_1^j$	$i+N_1^j+1$	$i+N_1^j+2$	$i+N_1^j+3$	$i+N_1^j+4$
Link (r_j, r_{j+1})				
a_i				
	b_{i+1}			
		$f^1(a_i, b_{i+1})$		
Link (r_{j+1}, r_{j+2})				
	b_{i+1}			
		a_i		
			$f^1(a_i, b_{i+1})$	
				$f^2(a_i, b_{i+1})$

dividing both sides by j, we have

$$k[(T - N_2^{L+1} + 1) + 1/j] \leq n(1 + 1/j)(T - N_1^{L+1} + 1) \tag{53}$$

Since (53) holds for all $j \in \mathbb{N}$, it follows that (21) hold.

APPENDIX C
EXAMPLES FOR RATE CHANGE IN RELAY

As mentioned above, relay r_j may need to increase or decrease the rate of the code used by relay r_{j-1}. Below, we show examples for the following two cases:

- $\frac{k}{n_{j+2}} > \frac{k}{n_{j+1}}$. This means that $n_{j+2} < n_{j+1}$, i.e., that the block size of the MDS code used by relay r_{j+1} is smaller than the block size used by relay r_j. At time $i+T-\sum_{j+1}^{L+1} N_l + 1$, node r_{j+1} can recover the original data and send any of the erased symbols of the code used by r_j.
  An example is given in Table VIII for $N_{j+1} = 2$, $N_{j+2} = 1$, $T' = 4$ (where $T' = T - \sum_{l=1, l\neq j+1, j+2} N_l$). We note that in this example $k = T - \sum N_l + 1 = T' - N_{j+1} - N_{j+2} = 2$.
  Relay r_{j+1} forwards the first $k - 1 = 1$ symbols it receives. At $i + N_1^j + 3$ the relay can recover the original data. Hence, from this point it sends (for example) the erased symbols.
- $\frac{k}{n_{j+2}} < \frac{k}{n_{j+1}}$. This means that $n_{j+2} > n_{j+1}$, i.e., the block size of the code used by relay r_{j+1} is larger than the block size used by relay r_j.
  At time $i+T-\sum_{j+1}^{L+1} N_l+1$, relay r_{j+1} can again recover the original data and hence transmit additional $n_{j+2} - k$ symbols needed to allow handling any N_{j+2} erasures in the link (r_{j+1}, r_{j+2}).
  An example is given in Table IX for $N_{j+1} = 1,\ N_{j+2} = 2,\ T' = 4$ (where, again, $T' = T - \sum_{l=1, l\neq j+1, j+2}$).
  Relay r_{j+1} forwards the first $k - 1 = 1$ symbols it receives. At $i + N_1^j + 2$, the relay can recover the original data. Hence, from this point is sends (for example) the erased symbols while adding parity symbols to reach the required rate.

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